Rational Geometry: A Textbook For The Science Of Space, Based On Hilbert's Foundations (1904)

David Hilbert
George Bruce Halsted

RATIONAL GEOMETRY

A TEXT-BOOK FOR

THE SCIENCE OF SPACE

BASED ON HILBERT'S FOUNDATIONS

BY

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PREFACE.

WRITING to Professor Hilbert my desire to base a text-book on his foundations, he answered: "Ueber Ihre Idee aus meinen Grundlagen eine Schul-Geometrie zu machen, bin ich sehr erfreut. Ich glaube auch, dass dieselben sich sehr gut dazu eignen werden."

Geometry at last made rigorous is also thereby made more simple.

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CONTENTS.

TABLE OF SYMBOLS.

We denote

triangle by $\triangle$; the vertices by A, B, C;
the angles at A, B, C by α, β, γ;
the opposite sides by a, b, c;
the altitudes from A, B, C by h_a, h_b, h_c;
the bisectors of α, β, γ by t_a, t_b, t_c;
the medians to a, b, c by m_a, m_b, m_c;
the feet of h_a, h_b, h_c by D, E, F;
the centroid by G;
the orthocenter by H;
the in-center by I; the in-radius by r;
the ex-centers beyond a, b, c by I_1, I_2, I_3; their ex-radii by r_1, r_2, r_3;
the circumcenter by O; the circumradius by R;
angle by $\measuredangle$; angles by $\measuredangle$s;
angle made by the rays BA and BC by $\measuredangle ABC$;
angle made by the rays a and b both from the point O by $\measuredangle(a, b)$ or $\measuredangle ab$;
bisector by bi'; circle by $\odot$; circles by $\odot$s;
circle with center C and radius r by $\odot C(r)$;
congruent by $\equiv$;
equal or equivalent by completion by $=$;
for example [*exempli gratia*] by *e.g.*
greater than by $>$;
less than by $<$; minus by $-$;
parallel by $\parallel$; parallels by $\parallel$s;

parallelogram by ||g'm; perimeter [sum of sides] by p;
perpendicular by $\perp$; perpendiculars by $\perp$s;
plus by $+$;
quadrilateral by quad';
right by r't;
spherical angle by $\widehat{\measuredangle}$;
spherical triangle by $\widehat{\triangle}$;
similar by $\sim$;
symmetrical by $\dotplus$;
therefore by $\therefore$

RATIONAL GEOMETRY.

CHAPTER I.

ASSOCIATION.

The Geometric Elements.

1. Geometry is the science created to give understanding and mastery of the external relations of things; to make easy the explanation and description of such relations and the transmission of this mastery.

2. Convention. We think three different sorts of things. The things of the first kind we call *points*, and designate them by A, B, C, . . . ; the things of the second system we call *straights*, and designate them by a, b, c, . . . ; the things of the third set we call *planes*, and designate them by α, β, γ, ...

3. We think the points, straights, and planes in certain mutual relations, and we designate these relations by words such as "lie," "between," "parallel," "congruent."

The exact and complete description of these rela-

tions is accomplished by means of the *assumptions* of geometry.

4. The assumptions of geometry separate into five groups. Each of these groups expresses certain connected fundamental postulates of our intuition.

I. The first group of assumptions: assumptions of association.

5. The assumptions of this group set up an association between the concepts above mentioned, points, straights, and planes. They are as follows:

I 1. *Two distinct points, A, B, always determine a straight, a.*

Of such points besides "determine" we also employ other turns of phrase; for example, A "lies on" a, A "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc.

When we say two things *determine* some other thing, we simply mean that if the two be given, then this third is explicitly and uniquely given.

If A lies on a and besides on another straight b we use also the expression: "the straights" a "and" b "have the point A in common."

I 2. ANY *two distinct points of a straight determine* THIS *straight; and on every straight there are at least two points.*

That is, if AB determine a and AC determine a, and B is not C, then also B and C determine a.

I 3. *Three points, A, B, C, not costraight, always determine a plane α.*

We use also the expressions:

A, B, C "lie in" α, A, B, C, "are points of" α, etc.

I 4. ANY *three non-costraight points A, B, C of a plane α determine* THIS *plane α.*

I 5. *If two points A, B of a straight a lie in a plane α, then every point of a lies in α.*

In this case we say: The straight a lies in α.

I 6. *If two planes α, β have a point A in common, then they have besides at least another point B in common.*

I 7. *In every plane there are at least three non-costraight points. There are at least four non-co-straight non-coplanar points.*

6. Theorem. *Two distinct straights cannot have two points in common.*

Proof. The two points being on the first straight determine (by I 2) that particular straight. If by hypothesis they are also on a second straight, therefore (by I 2) they determine this second straight. Therefore the first straight is identical with the second.

7. Theorem. *Two straights have one or no point in common.*

Proof. By 6 they cannot have two.

8. Theorem. *Two planes have no point or a straight in common.*

Proof. If they have one point in common, then (by I 6) they have a second point in common, and therefore (by I 5) each has in it the straight which (by I 1) is determined by these two points.

9. Corollary to 8. A point common to two planes

lies in a straight common to the two, which may be called their straight of intersection or their *meet*.

10. Theorem. *A plane and a straight not lying in it have no point or one point in common.*

Proof. If they had two points in common the straight would be (by I 5) situated completely in the plane.

11. Theorem. *Through a straight and a point not on it there is always one and only one plane.*

Proof. On the straight there are (by I 2) two points. These two with the point not on the straight determine (by I 3) a plane, in which (by I 5) they and the given straight lie. Any plane on this point and straight would be on the three points already used, hence (by I 4) identical with the plane determined.

12. Theorem. *Through two different straights with a common point there is always one and only one plane.*

Proof. Each straight has on it (by I 2) one point besides the common point, and (by 6) these two points are not the same point, and (by I 2) the three points are not costraight.

These three points determine (by I 3) a plane in which (by I 5) each of the two straights lies. Any plane on these straights would be on the three points already used, hence (by I 4) identical with the plane determined.

CHAPTER II.

BETWEENNESS.

II. The second group of assumptions: assumptions of betweenness.

13. The assumptions of this group make precise the idea "between," and make possible on the basis of this idea the *arrangement* of points.

14. Convention. The points of a straight stand in certain relations to one another, to describe which especially the word "*between*" serves us.

II 1. *If A, B, C are points of a straight, and B lies between A and C, then B also lies between C and A, and is neither C nor A.*

FIG. 1.

II 2. *If A and C are two points of a straight, then there is always at least one point B, which lies between*

FIG. 2.

A and C, and at least one point D, such that C lies between A and D.

II 3. *Of any three points of a straight there is always one and only one which lies between the other two.*

15. Definition. Two points A and B, upon a straight a, we call a segment or *sect*, and designate it with AB or BA. The points between A and B are said to be points of the sect AB or also situated *within* the sect AB. All remaining points of the straight a are said to be situated *without* the sect AB. The points A, B are called *end-points* of the sect AB.

II 4. (Pasch's assumption.) *Let A, B, C be three points not costraight and a a straight in the plane ABC going through none of the points A, B, C; if*

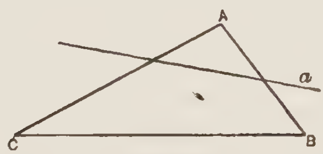

FIG. 3.

then the straight a goes through a point within the sect AB, it must always go either through a point of the sect BC or through a point of the sect AC.

Deductions from the assumptions of association and betweenness.

16. Theorem. *Between any two points of a straight there are always indefinitely many points.*

[Here taken for granted, and its proof removed to Appendix I.]

17. Theorem. If any finite number of points of

a straight are given, then they can always be arranged in a succession $A, B, C, D, E, \ldots, K$, such that B lies between A on the one hand and $C, D, E, \ldots, K$ on the other, further C between A, B on the one hand and $D, E, \ldots, K$ on the other, then D between A, B, C on the one hand and $E, \ldots, K$ on the other, and so on.

Besides this distribution there is only one other, the reversed arrangement, which is of the same character.

[This theorem is here taken for granted, and its proof removed to Appendix I.]

21. Theorem. *If A, B, C be not costraight, any straight in the plane ABC which has a point within the sect AB and a point within AC cannot have a point within BC.*

Proof. Suppose F, G, H three such costraight points.

One, say G, on AB, must (by II 3) lie between the others. Then the straight AB must (by II 4) have a point within the sect FC or the sect CH, which (by 7 and II 3) is impossible.

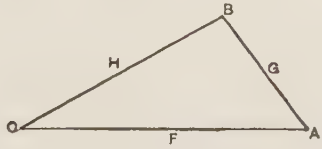

FIG. 4.

22. Theorem. *Every straight a, which lies in a plane α, separates the other points of this plane α into two regions, of the following character: every*

point A of the one region determines with every point B of the other region a sect AB, within which lies a point of the straight a; on the contrary, any two

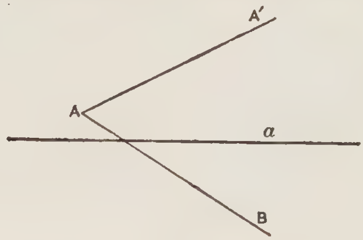

FIG. 5.

points A, A′ of one and the same region always determine a sect AA′ which contains no point of a.

Proof. Let A be a point of α which does not lie on a. Then reckon to one region all points P of the property, that between A and P, therefore

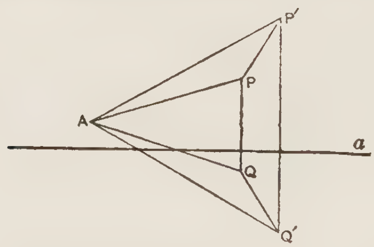

FIG. 6.

within AP, lies no point of a; to the other region all points Q such that within AQ lies a point of a.

Now is to be shown:

(1) On PP' lies no point of a.

(2) On QQ' lies no point of a.

(3) On PQ lies always a point of a.

(1) From hypothesis neither within AP nor AP' lies a point of a. This would contradict II 4, if within PP' were a point of a.

(2) By hypothesis there lies within AQ a point of a, likewise within AQ'; therefore (by 21) none within QQ'.

(3) By hypothesis AP contains no point of a; AQ on the other hand contains one such. Therefore (by II 4) a meets PQ.

23. Convention. If A, A', O, B are four costraight points such that O is between A and B but not between A and A'; then we say: the points A, A'

Fig. 7.

lie *in the straight a on one and the same side of the point O*, and the points A, B lie *in the straight a on different sides of the point O.*

24. Definition. The assemblage, aggregate, or totality of all points of the straight a situated on one and the same side of O is called a *ray* starting from O.

Consequently every point of a straight is the origin of two rays.

25. Convention. Using the notation of 22, we say: the points P, P' lie *in the plane α on one and the same side of the straight a* and the points P, Q lie *in the plane α on different sides of the straight a.*

26. Theorem. *Every two intersecting straights a, b separate the points of their plane α not on either into four regions such that if the end-points of a sect are both in one of these regions, the sect contains no point of either straight.*

Proof. Let O be their common point and A another point on b, and B another point on a. Then two points both on the A side of a and the B side of b make a sect which (by 22) can contain no point either of a or of b. So also if both were

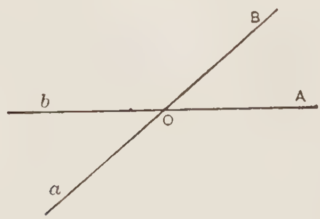

FIG. 8.

on the A side of a and the non-B side of b; or both on the non-A side of a and the B side of b; or both on the non-A side of a and the non-B side of b.

27. Definition. A system of sects AB, BC, CD, . . . , KL is called a *sect-train*, which joins the points A and L with one another. This sect-train will also be designated for brevity by $ABCD \ldots KL$.

The points within the sects AB, BC, CD, . . . , KL, together with the points A, B, C, D, . . . , K, L are all together called the *points of the sect-train.*

In particular if the point L is identical with the point A, then the sect-train is called a *polygon* and is designated as polygon $ABCD \ldots K$.

The sects AB, BC, CD, . . . , KA are called the *sides of the polygon.* The points A, B, C, D, . . . , K are called the *vertices of the polygon.*

A sect not a side but whose end-points are vertices is called a *diagonal* of the polygon.

Polygons with 3, 4, 5, . . . , n vertices are called *respectively triangles, quadrilaterals, pentagons, . . . , n-gons.*

28. If the vertices of a polygon are all distinct from one another and no vertex of the polygon falls within a side and finally no two sides of the polygon have a point within in common, then the polygon is called *simple.*

By quadrilateral is meant simple quadrilateral.

A *plane* polygon is one all of whose sides are coplanar.

A *convex* polygon is one no points of which are on different sides of the straight of any of its sides.

29. Theorem. Every simple polygon, whose vertices all lie in a plane α, separates the points of this plane α, which do not pertain to the sect-train of the polygon, into two regions, an inner and an outer, of the following character: if A is a point of the inner (*interior* point) and B a point of the outer (*exterior* point), then every sect-train which joins A with B has at least one point in common with the polygon; on the contrary if A, A' are two points of the inner and B, B' two points of the outer, then there are always sect-trains, which join A with A' and B with B' and have no point in common with the polygon.

There are straights in α which lie wholly outside the polygon; on the contrary no such straights which lie wholly within the polygon.

Proof. Any simple polygon by joining its vertices gives a number of triangles. For a triangle ABC there is (by 26) a region with points on the

A side of BC, the B side of CA, and the C side of AB, i.e., an inner region. Moreover, the straight determined by a point on b and a point on c both in non-A lies wholly without the region ABC, since it cannot again meet b or c and so cannot (by II 4)

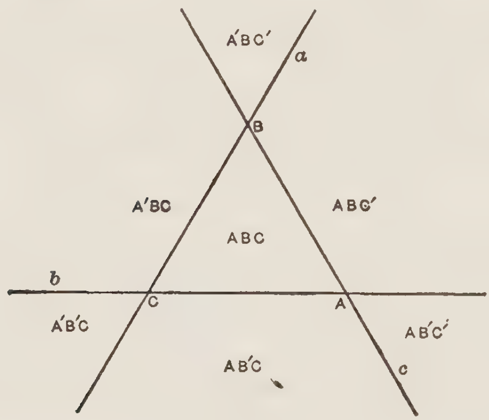

FIG. 9.

have a point in common with BC. Moreover, if any straight has a point within ABC, it has a point on a side. For the straight determined by the point within and any point on a side has (by II 4) a point on another side, thus making another triangle, in common with *one* side of which the given straight has a point, and therefore (by II 4) with *another* side, that is with a side of the original triangle.

30. Corollary to 29. A straight through a vertex and a point within a triangle has a point within the 'opposite' side.

31. Theorem. Every plane α separates all points not on it into two regions of the following character: every point A of the one region determines with every point B of the other region a sect AB, within which lies a point of α; on the contrary any two points A and A' of one and the same region always determine a sect AA', which contains no point of α.

Proof. Let A be a point which does not lie on α. Then reckon to the one region all points P of the property, that between A and P, therefore within AP, lies no point of α; to the other region all points Q such that within AQ lies a point of α.

Now is to be shown:

(1) On PP' lies no point of α.

(2) On QQ' lies no point of α.

(3) On PQ lies always a point of α.

(1) From hypothesis neither within AP nor AP' lies a point of α. Suppose now a point of α lay on PP'. Then the plane α and the plane APP' would have in common this point and consequently (by 9) a straight a. This straight goes through none of the points A, P, P'; it cuts PP'; it must therefore (by II 4) cut either AP or AP', which is contrary to hypothesis.

(2) By hypothesis there lies within AQ a point of α, likewise within AQ'. The intersection straight of the planes α and AQQ' therefore meets two sides of the triangle AQQ'; consequently (by 21) it cannot also meet the other side QQ'.

(3) AP contains by hypothesis no point of α; AQ on the other hand contains one such. The intersection straight of the planes α and APQ therefore

meets the side AQ and does not meet the side AP in triangle APQ. Therefore (by II 4) it meets the side PQ.

32. Convention. Using the notation of 31, we say: the points A, A' lie *on one and the same side of the plane* α, and the points A, B lie *on different sides of the plane* α.

Ex. 1. A straight cannot traverse more than 4 of the 7 regions of the plane determined by the straights of the sides of a triangle.

Ex. 2. Four coplanar straights crossing two and two determine 6 points. Choosing 4 as vertices we can get two convex quadrilaterals, one of which has its sides on the straights.

Ex. 3. Each vertex of an n-gon determines with the others $(n-1)$ straights. So together they determine $n(n-1)/2$.

Ex. 4. How many diagonals in a polygon of n sides.

Ex. 5. What polygon has as many diagonals as sides?

CHAPTER III.

CONGRUENCE.

III. The third group of assumptions: assumptions of congruence.

33. The assumptions of this group make precise the idea of congruence.

34. Convention. Sects stand in certain relations to one another, for whose description the word *congruent* especially serves us.

III 1. *If A, B are two points on a straight a, and A' a point on the same or another straight a', then we can find on the straight a' on a given ray from A' always one and only one point B' such that the sect AB is congruent to the sect $A'B'$.*

We write this in symbols $AB \equiv A'B'$.

Every sect is congruent to itself, i.e., always $AB \equiv AB$. *The sect AB is always congruent to the sect BA*, i.e., $AB \equiv BA$.

We also say more briefly, that every sect can be taken on a given side of a given point on a given straight in one and only one way.

III 2. *If a sect AB is congruent as well to the sect $A'B'$ as also to the sect $A''B''$, then is also $A'B'$ con-*

gruent to the sect $A''B''$, i.e., if $AB \equiv A'B'$ and $AB \equiv A''B''$, then is also $A'B' \equiv A''B''$.

III 3. *On the straight a let* AB *and* BC *be two sects without common points, and furthermore* $A'B'$ *and* $B'C'$ *two sects on the same or another straight, likewise without common points; if then* $AB \equiv A'B'$ *and* $BC \equiv B'C'$, *so always also* $AC \equiv A'C'$.

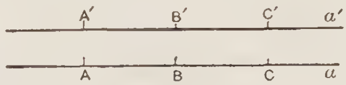

FIG. 10.

35. Definition. Let α be any plane and h, k any two distinct rays in α going out from a point O, and pertaining to *different* straights. These two rays h, k we call an *angle*, and designate it by $\sphericalangle (h, k)$ or $\sphericalangle (k, h)$. The rays h and k, together with the point O, separate the other points of the plane α into two regions of the following character: if A is a point of the one region and B of the other region, then every sect-train which joins A with B, goes either through O or has with h or k at least one point in common; on the contrary if A, A' are points of the same region, then there is always a sect-train which joins A with A' and neither goes through O nor through a point of the rays h, k.

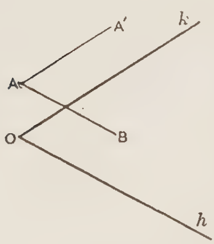

FIG. 11.

One of these two regions is distinguished from the other because each sect which joins any two points of this distinguished region always lies wholly

in it; this distinguished region is called the *interior* of the angle (h, k) in contradistinction from the other region, which is called the *exterior* of the angle (h, k). The interior of $\sphericalangle (h, k)$ is wholly on the same side of the straight h as is the ray k, and altogether on the same side of the straight k as is the ray h.

The rays h, k are called *sides* of the angle, and the point O is called the *vertex* of the angle.

III 4. *Given any angle (h, k) in a plane α and a straight a' in a plane α', also a determined side of a' on α'. Designate by h' a ray of the straight a' starting from the point O'; then there is in the plane α'* ONE AND ONLY ONE *ray k' such that the angle (h, k) is congruent to the angle (h', k'), and likewise all interior points of the angle (h', k') lie on the given side of a'.*

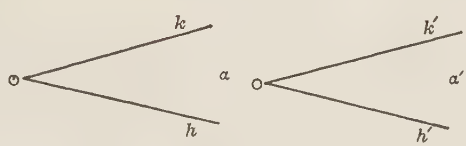

FIG. 12

In symbols:

$$\sphericalangle (h, k) \equiv \sphericalangle (h', k').$$

Every angle is congruent to itself, i.e., always

$$\sphericalangle (h, k) \equiv \sphericalangle (h, k).$$

The angle (h, k) is always congruent to the angle (k, h), i.e., $\sphericalangle (h, k) \equiv \sphericalangle (k, h)$.

We say also briefly, that in a given plane every angle can be set off towards a given side against a

given ray, but in a uniquely determined way. There is one and only *one* such angle congruent to a given angle. We say an angle so taken is uniquely determined.

III 5. *If an angle (h, k) is congruent as well to the angle (h', k') as also to the angle (h'', k''), then is also the angle (h', k') congruent to the angle (h'', k'')*; i.e., if $\measuredangle(h, k) \equiv \measuredangle(h', k')$ and $\measuredangle(h, k) \equiv \measuredangle(h'', k'')$, then always $\measuredangle(h', k') \equiv \measuredangle(h'', k'')$.

36. Convention. Let ABC be any assigned triangle; we designate the two rays going out from A through B and C respectively by h and k. Then the angle (h, k) is called the angle of the triangle ABC included by the sides AB and AC or opposite the side BC. It contains in its interior all the inner points of the triangle ABC and is designated by $\measuredangle BAC$ or $\measuredangle A$.

III 6. *If for two triangles ABC and $A'B'C'$ we have the congruences*

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \measuredangle BAC \equiv \measuredangle B'A'C',$$

then always are fulfilled the congruences

$$\measuredangle ABC \equiv \measuredangle A'B'C' \quad \text{and} \quad \measuredangle ACB \equiv \measuredangle A'C'B'.$$

Deductions from the assumptions of congruence.

37. Convention. Suppose the sect AB congruent to the sect $A'B'$. Since, by assumption III 1, also the sect AB is congruent to AB, so follows from III 2 that $A'B'$ is congruent to AB; we say: the two sects AB and $A'B'$ are *congruent to one another*.

38. Convention. Suppose $\measuredangle(h, k) \equiv \measuredangle(h', k')$.

Since (by III 4) $\sphericalangle(h, k) \equiv \sphericalangle(h, k)$, therefore (by III 5) $\sphericalangle(h', k') \equiv \sphericalangle(h, k)$. We say then: the two angles $\sphericalangle(h, k)$ and $\sphericalangle(h', k')$ are *congruent to one another*.

39. Definition. Two angles having the same vertex and one side in common, while the sides not common form a straight, are called *adjacent angles*.

40. Definition. Two angles with a common vertex and whose sides form two straights are called *vertical angles*.

41. Definition. Any angle which is congruent to one of its adjacent angles is called a *right angle*.

Two straights which make a right angle are said to be *perpendicular* to one another.

42. Convention. Two triangles ABC and $A'B'C'$ are called *congruent* to one another, if all the congruences

$$AB \equiv A'B', \quad AC \equiv A'C', \quad BC \equiv B'C',$$
$$\sphericalangle A \equiv \sphericalangle A', \quad \sphericalangle B \equiv \sphericalangle B', \quad \sphericalangle C \equiv \sphericalangle C'$$

are fulfilled.

43. (First congruence theorem for triangles.)

Triangles are congruent if they have two sides and the included angle congruent.

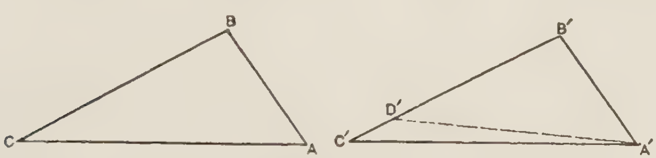

Fig. 13.

In the triangles ABC and $A'B'C'$ take $AB \equiv A'B'$, $AC \equiv A'C'$, $\sphericalangle A \equiv \sphericalangle A'$.

To prove $\triangle ABC \equiv \triangle A'B'C'$.

Proof. By assumption III 6 the congruences $\sphericalangle B \equiv \sphericalangle B'$ and $\sphericalangle C \equiv \sphericalangle C'$ are fulfilled, and so we have only to show that the sides BC and $B'C'$ are congruent to one another.

Suppose now, on the contrary, that BC were not congruent to $B'C'$, and take on ray $B'C'$ (by III 1) the point D', such that $BC \equiv B'D'$. Then the two triangles ABC and $A'B'D'$ will have, since $\sphericalangle B \equiv \sphericalangle B'$, two sides and the included angle respectively congruent; by assumption III 6, consequently, are in particular the two angles BAC and $B'A'D'$ congruent to one another. By assumption III 5, consequently, must therefore also the two angles $B'A'C'$ and $B'A'D'$ be congruent to one another. This is impossible, since, by assumption III 4, against a given ray toward a given side in a given plane there is only one angle congruent to a given angle. So the theorem is completely established.

44. (Second congruence theorem for triangles.)

Two triangles are congruent if a side and the two adjoining angles are respectively congruent.

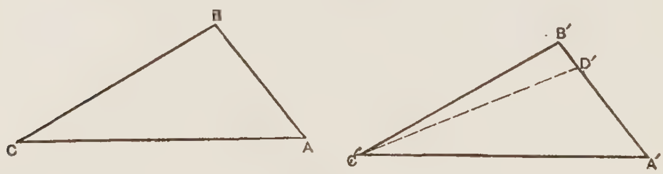

FIG. 14.

In the triangles ABC and $A'B'C'$ take $AC \equiv A'C'$, $\sphericalangle A \equiv \sphericalangle A'$, $\sphericalangle C \equiv \sphericalangle C'$.

To prove $\triangle ABC \equiv \triangle A'B'C'$.

Proof. Suppose now, on the contrary, AB is not $\equiv A'B'$, and take on ray $A'B'$ the point D', such that $AB \equiv A'D'$. By III 6, $\measuredangle ACB \equiv \measuredangle A'C'D'$, but by hypothesis $\measuredangle ACB \equiv \measuredangle A'C'B'$. Therefore (by III 5) $\measuredangle A'C'B' \equiv \measuredangle A'C'D'$. But this is impossible, since (by III 4) in a given plane against a given ray toward a given side there is only *one* angle congruent to a given angle.

Consequently our supposition, AB not $\equiv A'B'$, is false, and so $AB \equiv A'B'$.

Now follows (by 43) that $\triangle ABC \equiv \triangle A'B'C'$.

45. Theorem. *If two angles are congruent, so are also their adjacent angles.*

Take $\measuredangle ABC \equiv \measuredangle A'B'C'$.

To prove $\measuredangle CBD \equiv \measuredangle C'B'D'$.

Proof. Choose the points A', C', D' on the sides from B' so that $A'B' \equiv AB$, $C'B' \equiv CB$, $DB \equiv D'B'$.

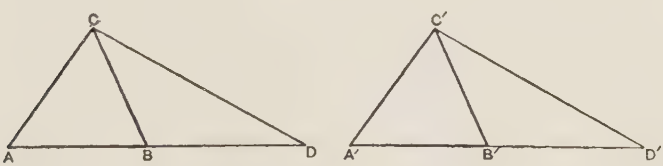

Fig. 15.

In the two triangles ABC and $A'B'C'$ then the sides AB and CB are congruent respectively to the sides $A'B'$ and $C'B'$, and since moreover the angles included by these sides are congruent by hypothesis, so follows (by 43) the congruence of those triangles, that is, we have the congruences

$$AC \equiv A'C' \quad \text{and} \quad \measuredangle BAC \equiv \measuredangle B'A'C'.$$

Now since (by III 3) sect $AD \equiv A'D'$, so follows (again by 43) the congruence of the triangles CAD and $C'A'D'$, that is, we have the congruences $CD \equiv C'D'$ and $\measuredangle ADC \equiv \measuredangle A'D'C'$, and hence follows, through consideration of the triangles BCD and $B'C'D'$ (by III 6), the congruence of the angles CBD and $C'B'D'$.

46. Theorem. *Vertical angles are congruent.*

Proof. By III 4, $\measuredangle ABC \equiv \measuredangle CBA$. Therefore, by 45, their adjacent angles are congruent, $\measuredangle CBD \equiv \measuredangle ABF$.

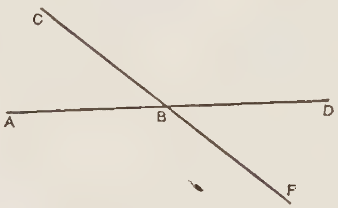

FIG. 16.

47. Theorem. *Through a point A, not on a straight a, there is one and only one perpendicular to a.*

Proof. Take any two points P, Q on a. Take from P against the side PQ of $\measuredangle APQ$, and on the non-A side of a, $\measuredangle BPQ \equiv \measuredangle APQ$. Take $PB \equiv PA$. Since A and B lie on different sides of a, there must be a point O of sect AB on a. Then AOB is perpendicular to a.

For (by 43) $\triangle BPO \equiv \triangle APO$, so $\measuredangle BOP \equiv \measuredangle AOP$. But these are adjacent. Therefore, by definition 41, AOP is a right angle.

Moreover this perpendicular is unique. For suppose any straight AO' perpendicular to a at O', and

take on this straight on the non-A side of a the sect $O'B' \equiv O'A$. Then from hypothesis $\sphericalangle PO'B' \equiv \sphericalangle PO'A$ and so (by 43) $\triangle PO'B' \equiv \triangle PO'A$. There-

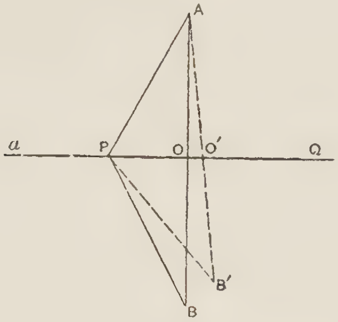

FIG. 17.

fore $\sphericalangle B'PO' \equiv \sphericalangle APO'$ and $B'P \equiv AP$. Therefore (by III 5 and III 2), $\sphericalangle B'PO' \equiv \sphericalangle BPO$ and $B'P \equiv BP$. Hence the points B and B' are not different. Therefore no second perpendicular from A to a can exist.

48. Theorem. Let the angle (h, k) in the plane α be congruent to the angle (h', k') in the plane α', and further let l be a ray of the plane α, which goes out from the vertex of the angle (h, k) and lies in the interior of this angle; then there is always a ray l' in the plane α', which goes out from the vertex of the angle (h', k') and lies in the interior of this angle, such that $\sphericalangle (h, l) \equiv \sphericalangle (h', l')$ and $\sphericalangle (k, l) \equiv \sphericalangle (k', l')$.

Proof. Designate the vertex of $\sphericalangle (h, k)$ by O, and the vertex of $\sphericalangle (h', k')$ by O', and then determine on the sides h, k, h', k', the points A, B, A', B', so that we have the congruences

$$OA \equiv O'A' \quad \text{and} \quad OB \equiv O'B'.$$

Because of the congruence of the triangles OAB and $O'A'B'$ (by 43)

$AB \equiv A'B'$, $\sphericalangle OAB \equiv \sphericalangle O'A'B'$, $\sphericalangle OBA \equiv \sphericalangle O'B'A'$.

The straight AB (by 30) cuts l, say in C; then we determine on the sect $A'B'$ the point C', such that $A'C' \equiv AC$, then is $O'C'$ the ray sought, l'.

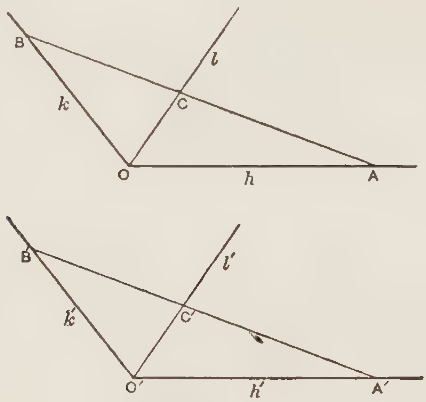

FIG. 18.

In fact, from $AC \equiv A'C'$ and $AB \equiv A'B'$ we may, by means of III 3, deduce the congruence $BC \equiv B'C'$. Therefore (by III 6) $\sphericalangle AOC \equiv \sphericalangle A'O'C'$ and $\sphericalangle BOC \equiv \sphericalangle B'O'C'$.

49. Theorem. Let h, k, l on the one hand and h', k', l' on the other each be three rays going out from a point and lying in a plane; if then we have the congruences $\sphericalangle (h, l) \equiv \sphericalangle (h', l')$ and $\sphericalangle (k, l) \equiv \sphericalangle (k', l')$, then also is always

$$\sphericalangle (h, k) \equiv \sphericalangle (h', k').$$

Proof. The rays are supposed such that either no point is interior to $\measuredangle(h, l)$ and $\measuredangle(k, l)$, or to $\measuredangle(h', l')$ and $\measuredangle(k', l')$, or else that if one of these angles be within a second, then the angle congruent to the first is within the fourth.

I. In the first case, if l be supposed *within* $\measuredangle(h, k)$, take against h' toward k', $\measuredangle(h', k'') \equiv \measuredangle(h, k)$. By

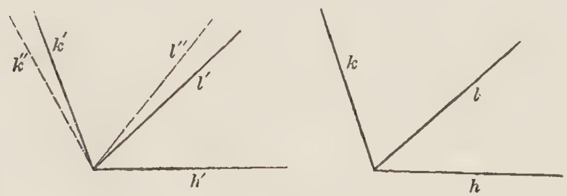

FIG. 19.

48, take in angle (h', k'') ray l'' such that $\measuredangle(h', l'') \equiv \measuredangle(h, l)$ and $\measuredangle(l'', k'') \equiv \measuredangle(l, k)$. But by hypothesis $\measuredangle(h', l') \equiv \measuredangle(h, l)$. Therefore, by III 5, $\measuredangle(h', l') \equiv \measuredangle(h', l'')$, and so, by III 4, ray l'' is identical with ray l'. Then $\measuredangle(k'', l'') \equiv \measuredangle(k'', l') \equiv \measuredangle(k, l) \equiv \measuredangle(k', l')$. So $\measuredangle(k'', l') \equiv \measuredangle(k', l')$, and, by III 4, ray k'' is identical with ray k'.

But $\measuredangle(h', k'') \equiv \measuredangle(h, k)$. Therefore $\measuredangle(h, k) \equiv \measuredangle(h', k')$.

If, however, l be supposed *not within* $\measuredangle(h, k)$, then it will lie in $\measuredangle(h'', k'')$ vertical to $\measuredangle(h, k)$. For it cannot lie in $\measuredangle(h, k'')$ adjacent to $\measuredangle(h, k)$, since then $\measuredangle(l, k)$ would contain $\measuredangle(h, l)$, contradicting the hypothesis in this case of no point interior to these two given angles. For like reason it cannot lie in

$\measuredangle (h'', k)$ adjacent to $\measuredangle (h, k)$, since then $\measuredangle (h, l)$ would contain $\measuredangle (l, k)$. Thus the ray m costraight with l is within $\measuredangle (h, k)$, and m' costraight with l' is within $\measuredangle (h', k')$.

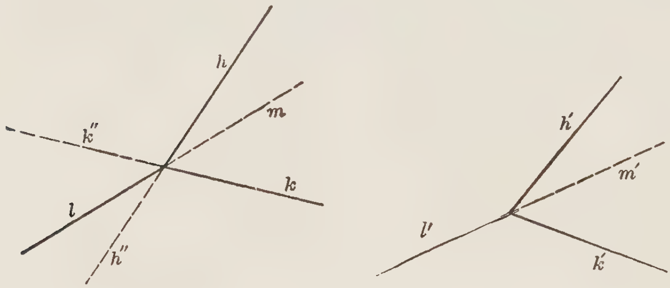

FIG. 20.

Then (by 45) $\measuredangle (h, m) \equiv \measuredangle (h', m')$ and $\measuredangle (k, m) \equiv \measuredangle (k', m')$ [$\measuredangle$'s adjacent to congruent $\measuredangle$'s are congruent], and so this sub-case is reduced to the preceding.

II. The remaining case, where one angle $\measuredangle (h, l)$ is within another, $\measuredangle (k, l)$, follows at once from 48.

51. Theorem. *All right angles are congruent.* Let angle BAD be congruent to its adjacent angle CAD, and likewise let the angle $B'A'D'$ be congruent to its adjacent angle $C'A'D'$; then are $\measuredangle BAD$, $\measuredangle CAD$, $\measuredangle B'A'D'$, $\measuredangle C'A'D'$ all right angles.

To prove $\measuredangle BAD \equiv \measuredangle B'A'D'$.

Proof. Suppose, contrary to our proposition, the right angle $B'A'D'$ were not congruent to the right angle BAD, and then set off $\measuredangle B'A'D'$ against ray

AB so that the resulting side AD'' falls either in the interior of the angle BAD or of the angle CAD; suppose we have the first of these cases.

Because $\measuredangle B'A'D' \equiv \measuredangle BAD''$, therefore, by 45, $\measuredangle C'A'D' \equiv \measuredangle CAD''$; and since by hypothesis $\measuredangle B'A'D' \equiv \measuredangle C'A'D'$, therefore, by III 5, $\measuredangle BAD'' \equiv \measuredangle CAD''$. Since further $\measuredangle BAD$ is congruent to $\measuredangle CAD$, so there is (by 48) within the angle CAD a ray AD''' such that

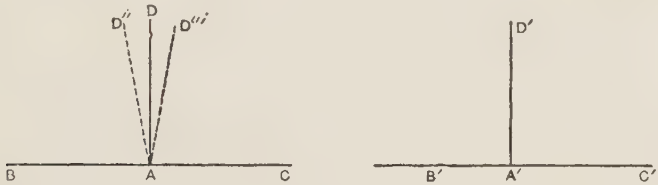

FIG. 21.

$\measuredangle BAD'' \equiv \measuredangle CAD'''$ and also $\measuredangle DAD'' \equiv \measuredangle DAD'''$. But we had $\measuredangle BAD'' \equiv \measuredangle CAD''$, and therefore we must (by III 5) also have $\measuredangle CAD'' \equiv \measuredangle CAD'''$. This is impossible, since (by III 4) every angle can be set off against a given ray toward a given side in a given plane only in *one* way.

Herewith is the proof for the congruence of right angles completed.

52. Corollary to 51. At a point A of a straight a there is not more than one perpendicular to a.

53. Definition. When any two angles are congruent to two adjacent angles, each is said to be the *supplement* of the other.

54. Definition. If any angle can be set off against one of the rays of a right angle so that its second

side lies within the right angle, it is called an *acute* angle.

55. Definition. Any angle neither right nor acute is called an *obtuse* angle.

56. Definition. A triangle with two sides congruent is called an *isosceles* triangle.

57. Theorem. *The angles opposite the congruent sides of an isosceles triangle are congruent.*

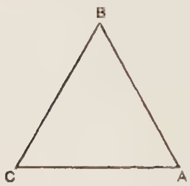

Fig. 22.

Let ABC be an isosceles triangle, having $AB \equiv BC$.

To prove $\measuredangle A \equiv \measuredangle C$.

Proof. Since in the triangles ABC and CBA we have the congruences $AB \equiv CB$, $BC \equiv BA$, $\measuredangle ABC \equiv \measuredangle CBA$, therefore (by III 6) $\measuredangle CAB \equiv \measuredangle ACB$.

58. (Third congruence theorem for triangles.) *Two triangles are congruent if the three sides of the one are congruent, respectively, to the three sides of the other.*

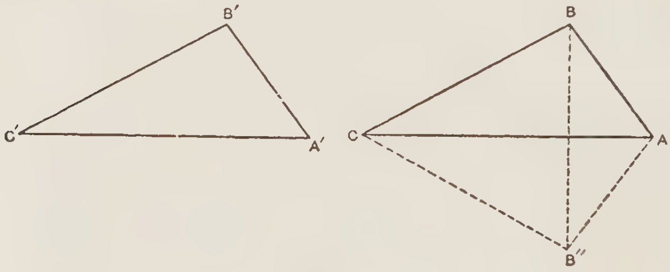

Fig. 23.

In the triangles ABC and $A'B'C'$ take $AB \equiv A'B'$, $AC \equiv A'C'$, $BC \equiv B'C'$.

To prove $\triangle ABC \equiv \triangle A'B'C'$.

Proof. In the plane of ABC toward the side of the straight AC not containing B against the ray AC take the angle $CAB'' \equiv C'A'B'$. Take the sect $AB'' \equiv A'B'$. Then (by 43) $\triangle AB''C \equiv \triangle A'B'C'$. Therefore $B''C \equiv BC$, and $\triangle BCB''$ is isosceles; therefore (by 57) $\measuredangle CBB'' \equiv \measuredangle CB''B$. So also is $\triangle BAB''$ isosceles and $\therefore$ $\measuredangle ABB'' \equiv \measuredangle AB''B$. Therefore (by 49) the angle $ABC \equiv \measuredangle AB''C$. But $\measuredangle AB''C \equiv \measuredangle A'B'C'$. $\therefore$ (by 43) $\triangle ABC \equiv \triangle A'B'C'$.

59. If A, B, C be any three points not costraight, then (by the method used in 58) we can construct a point B'' such that $AB'' \equiv AB$ and $CB'' \equiv CB$.

Therefore a point D such that no other point whatsoever, say D'', gives $AD'' \equiv AD$ and $CD'' \equiv CD$, must be costraight with AC.

The following have been given as definitions:

If A and B are two distinct points, the straight AB is the aggregate of points P for none of which is there any point Q such that $QA \equiv PA$ and $QB \equiv PB$.

If A, B, C are distinct points not costraight, the plane ABC is the aggregate of points P for none of which is there any point Q such that $QA \equiv PA$, $QB \equiv PB$, and $QC \equiv PC$.

60. Convention. Any finite number of points is called a *figure;* if all points of the figure lie in a plane, it is called a *plane figure.*

61. Convention. Two figures are called *congruent* if their points can be so mated that the sects and angles in this way coupled are all congruent.

Congruent figures have the following properties:

If three points be costraight in any one figure their mated points are also, in every congruent figure, costraight. The distribution of points in corresponding planes in relation to corresponding straights is in congruent figures the same; the like holds for the order of succession of corresponding points in corresponding straights.

62. The most general theorem of congruence for the plane and in general is expressed as follows:

If $(A, B, C, \ldots)$ and $(A', B', C', \ldots)$ are congruent plane figures and P denotes a point in the plane of the first, then we can always find in the plane of the second figure a point P' such that $(A, B, C, \ldots, P)$ and $(A', B', C', \ldots, P')$ are again congruent figures.

If each of the figures contains at least three non-costraight points, then is the construction of P' only possible in *one* way.

If $(A, B, C, \ldots)$ and $(A', B', C', \ldots)$ are congruent figures and P any point whatsoever, then we can always find a point P', such that the figures $(A, B, C, \ldots, P)$ and $(A', B', C', \ldots, P')$ are congruent.

If the figure $(A, B, C, \ldots)$ contains at least four non-coplanar points, then the construction of P' is only possible in *one* way.

This theorem contains the weighty result, that all facts of congruence are exclusively consequences (in association with the assumption-groups I and II) of the six assumptions of congruence already above set forth.

This theorem expresses the existence of a certain reversible unique transformation of the aggre-

gate of all points into itself with which we are familiar under the name of *motion* or displacement.

We have here founded the idea of motion upon the congruence assumptions. Thereby we have based the idea of motion on the congruence idea.

The inverse way, to try to prove the congruence assumptions and theorems with help of the motion idea, is false and fallacious, since the intuition of rigid motion involves, contains, and uses the congruence idea.

63. Exercises.

Ex. 6. Show a number of cases where two straights determine a point. Show cases where two straights do not determine a point. Are any of these latter pairs coplanar?

Ex. 7. Show cases where three coplanar straights determine 3 points; 2 points; 1 point. Are there cases where they determine no point?

Ex. 8. How many straights are, in general, determined by 3 points? by 4 coplanar points? What special cases occur?

Ex. 9. Any part of a triangle together with the two adjoining parts determine the 3 other parts. Explain.

Ex. 10. Try to state the first two congruence theorems for triangles so that either can be obtained from the other by simply interchanging the words *side* and *angle*.

Ex. 11. Principle of Duality in the Plane.

In theorems of configuration and determination we may interchange point and straight, sect and angle. Try to write down a theorem of which the dual is true; is false.

Ex. 12. If two angles of a triangle are congruent it is isosceles.

Ex. 13. If the sides of a $\triangle$ are $\equiv$, so are the $\measuredangle$s. Dual?

Ex. 14. In an isosceles $\triangle$, sects to the sides from the ends of the base making with it $\equiv \measuredangle$s are $\equiv$.

Ex. 15. If any two sects from the ends of a side of a $\triangle$ to the other sides making $\equiv \measuredangle$s are $\equiv$, the $\triangle$ is isosceles.

64. Definition. Two *parallels* are coplanar straights with no common point.

65. No assumption about parallels is necessary for the establishment of the facts of congruence or motion.

66. Theorem. *Through a point A without a straight a there is always one parallel to a.*

Proof. Take the ray from the given point A through any point B of the straight a. Let C be any other point of the straight a. Then take in the plane ABC an angle congruent to $\measuredangle ABC$ against AB at the point A toward that side not containing C. The straight so obtained through A does not

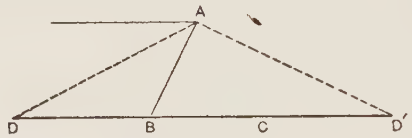

FIG. 24.

meet straight a. If we supposed it to cut a in the point D, and that, say, B lay between C and D, then we could take on a a point D', such that B lay between D and D', and moreover $AD \equiv BD'$. Because of the congruence of the triangles ABD and BAD' (by 43), therefore $\measuredangle ABD \equiv \measuredangle BAD'$; and since the angles ABD' and ABD are adjacent angles, so must then, having regard to 45, also the angles BAD and BAD' be adjacent angles. But because of 6, this is not the case.

67. Definition. A straight cutting across other straights is called a *transversal*.

68. Definition. If, in a plane, two straights are cut in two distinct points A, B by a transversal, at each of these points four angles are made. Of these eight, four, having each the sect AB on a side [e.g., 3, 4, 1′, 2′], are called *interior angles*. The other four are called *exterior angles*. Pairs of angles, one at each point, which lie on the same side of the transversal, the one exterior and the other interior, are called *corresponding angles* [e.g., 1 and 1′].

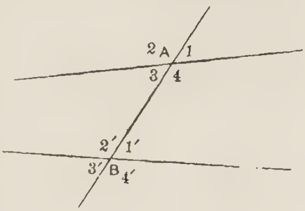

Fig. 25.

Two non-adjacent angles on opposite sides of the transversal, and both interior or both exterior, are called *alternate angles* [e.g., 3 and 1′].

Two angles on the same side of the transversal, and both interior or both exterior, are called *conjugate angles* [e.g., 4 and 1′].

69. Theorem. *Two coplanar straights are parallel if a transversal makes congruent alternate angles.* [Proved in 66.]

70. Theorem. If two straights cut by a transversal have corresponding angles congruent they are parallel.

Proof. The angle vertical to one is alternate to the other.

Ex. 16. If two corresponding or two alternate angles are congruent, or if two interior or two exterior angles on the same side of the transversal are supplemental,

then every angle is congruent to its corresponding and to its alternate angle, and is supplemental to the angle on the same side of the transversal which is interior or exterior according as the first is interior or exterior.

Ex. 17. If two interior or two exterior angles on the same side of the transversal are supplemental, the straights are parallel.

Ex. 18. Two straights perpendicular to the same straight are parallel.

Ex. 19. Construct a right angle.

Ex. 20. On the ray from the vertex of a triangle costraight with a side take a sect congruent to that side. The two new end-points determine a straight parallel to the triangle's third side.

Ex. 21. On one side of any $\measuredangle$ with vertex A take any two sects AB, AC and on the other side take congruent to these AB', AC'. Prove that BC' and $B'C$ intersect, say at D. Prove $BC' \equiv B'C$, $\triangle BCD \equiv \triangle B'C'D$, $\measuredangle BAD \equiv \measuredangle B'AD$.

Ex. 22. From two given points on the same side of a given st' find st's crossing on that given st', and making congruent $\measuredangle$'s with it.

Ex. 23. Construct a triangle, given the base, an angle at the base, and the sum of the other two sides [$\triangle$ from α, b, $a+c$].

Ex. 24. If the pairs of sides of a quadrilateral not consecutive are congruent, they are $\parallel$.

Ex. 25. On a given sect as base construct an isosceles $\triangle$.

Ex. 26. If on the sides AB, BC, CA of an equilateral $\triangle$, $AD \equiv BE \equiv CF$, then $\triangle DEF$ is equilateral, as is $\triangle$ made by AE, BF, CD.

CHAPTER IV.

PARALLELS.

IV. Assumption of Parallels (Euclid's Postulate).

IV. *Through a given point there is not more than one parallel to a given straight.*

71. The introduction of this assumption greatly simplifies the foundation and facilitates the construction of geometry.

72. Theorem. *Two straights parallel to a third are parallel.*

Proof. Were 1 and 2 not parallel, then there would be through their intersection point two parallels to 3, which is in contradiction to IV.

73. Theorem. *If a transversal cuts two parallels, the alternate angles are congruent.*

Proof. Were say $\measuredangle BAD$ not $\equiv \measuredangle ABC$, then we could through A (by III 4) take a straight making $\measuredangle BAD' \equiv \measuredangle ABC$ [D' and D on same side of AB], and so we would have (by 69) through A two parallels to a, in contradiction to IV.

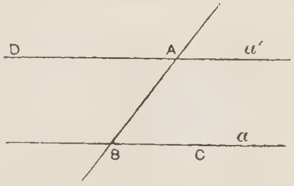

Fig. 26.

74. Corollary to 73. A perpendicular to one of two parallels is perpendicular to the other also.

75. Theorem. If a transversal cuts two parallels, the corresponding angles are congruent.

Proof. The angle vertical to one is alternate to the other.

Ex. 27. A straight meeting one of two parallels meets the other also.

Ex. 28. A straight cutting two parallels makes conjugate angles supplemental.

Ex. 29. If alternate or corresponding angles are unequal or if conjugate angles are not supplemental, then the straights meet. On which side of the transversal?

76. Theorem. A perpendicular to one of two parallels is parallel to a perpendicular to the other.

Proof. Either of the two given parallels makes (by 74) right angles with both perpendiculars, which therefore are parallel by 69.

77. Corollary to 76. Two straights respectively perpendicular to two intersecting straights cannot be parallel.

Proof. For if they were parallel, then (by 76) the intersecting straights would also be parallel.

78. Convention. When two angles are set off from the vertex of a third against its sides so that no point is interior to two, if the two sides not common are costraight, the three angles are said together to form two right angles.

79. *The angles of a triangle together form two right angles.*

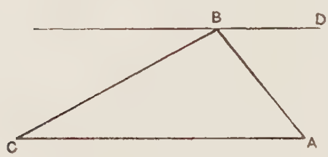

Fig. 27.

Proof. Take alternate $\measuredangle CBF \equiv \measuredangle C$ and $\measuredangle ABD \equiv \measuredangle A$; then (by 69) can neither BF nor BD cut AC. By the parallel postulate IV, then is FBD a straight.

80. Theorem. If two angles of one triangle are congruent to two of another, then the third angles are congruent.

Proof. Given $\measuredangle A \equiv \measuredangle A'$ and $\measuredangle B \equiv \measuredangle B'$. Take CP parallel (‖) to AB and $C'P' \parallel$ to $A'B'$. Then

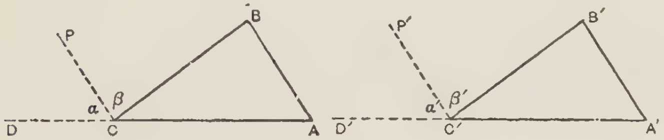

FIG. 28.

$\measuredangle \alpha \equiv \measuredangle A$ and $\measuredangle \beta \equiv \measuredangle B$, $\measuredangle \alpha' \equiv \measuredangle A'$ and $\measuredangle \beta' \equiv \measuredangle B'$. $\therefore$ (by 49) $\measuredangle BCD \equiv \measuredangle B'C'D'$. $\therefore$ (by 45) the adjacent angles $\measuredangle ACB \equiv \measuredangle A'C'B'$.

81. Theorem. *Two triangles are congruent if they have a side, an adjoining and the opposite angle respectively congruent.*

Proof. By 80 and 44.

Ex. 30. Every triangle has at least two acute angles.

Ex. 31. If the rays of one angle are parallel or perpendicular to those of another, the angles are congruent or supplemental.

Ex. 32. In a *right-angled* triangle [a triangle one of whose angles is a right angle] the two acute angles are complemental (calling two angles *complements* which together form a right angle).

82. Theorem. *In any sect AB there is always one and only one point C such that $AC \equiv BC$.*

Proof. Take any angle BAD at A against AB, and the angle congruent to it at B against BA and on the opposite side of a in the plane BAD; and take any sect AD on the free ray from A, and one

BF congruent to it on the free ray from B. The sect DF must cut a, say in C, since D and F are on opposite sides of a. Moreover, C is between A and B. Otherwise one of them, say A, would be between B and C. But then DA would have a point A on

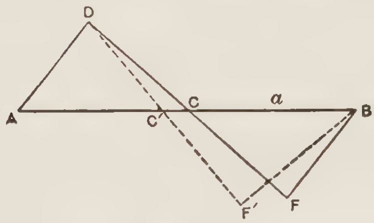

Fig. 29.

BC, a side of triangle FBC, and so (by II 4) must meet another side. But this is impossible, since it meets FC produced at D and is parallel to BF. Since thus $\measuredangle A \equiv \measuredangle B$, and $\measuredangle ACD \equiv \measuredangle BCF$ [vertical], therefore (by 81) $\triangle ACD \equiv \triangle BCF$.

Therefore $AC \equiv BC$.

If we suppose a second such point C', then on ray DC' take $C'F' \equiv DC'$. Therefore (by 43) $\measuredangle C'BF' \equiv \measuredangle DAC \equiv \measuredangle ABF$, and $BF' \equiv AD \equiv BF$. Therefore F' is F and C' is C.

83. Convention. The point C of the sect AB such that $AC \equiv BC$ may be called the *bisection-point* of AB, and to *bisect* AB shall mean to take this point C.

Ex. 33. Parallels through the end-points of a sect intercept congruent sects on any straight through its bisection-point.

Ex. 34. In a right-angled triangle the bisection-point of the *hypothenuse* (the side opposite the r't $\measuredangle$) makes equal sects with the three vertices.

Hint. Take one acute $\measuredangle$ in the r't $\measuredangle$.

84. Theorem. *Within any $\measuredangle (h, k)$ there is always one and only one ray, l, such that $\measuredangle (h, l) \equiv \measuredangle (k, l)$.*

Proof. From the vertex O take $OA \equiv OA'$. By 82 take C, the bisection-point of AA'. Then $\triangle AOC \equiv \triangle A'OC$ [$\triangle$s with 3 sides $\equiv$ are $\equiv$]. $\therefore \measuredangle AOC \equiv \measuredangle A'OC$.

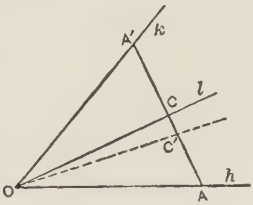

FIG. 30.

If we suppose a second such ray OC', then $\triangle AOC' \equiv \triangle A'OC'$ [$\triangle$s with 2 sides and the included $\measuredangle \equiv$ are $\equiv$]. $\therefore AC' \equiv A'C'$. $\therefore$ (by 82) C' is C.

85. Convention. The ray l of $\measuredangle (h, k)$ such that $\measuredangle (h, l) \equiv \measuredangle (l, k)$ may be called the *bisection ray* or bisector of $\measuredangle (h, k)$, and to *bisect* $\measuredangle (h, k)$ shall mean to take this ray l.

Ex. 35. An angle may be separated into 2, 4, 8, 16, . . . , 2^n congruent angles.

SYMMETRY.

86. Definition. Two points are said to be *symmetrical* with regard to a straight, when it bisects at right angles their sect. The straight is called their *axis of symmetry*. Two points have always one, and only one, symmetry axis.

FIG. 31.

A point has, with regard to a given axis of symmetry, always one, and only one, symmetrical point, namely, the one which ends the sect from the given point perpendicular to the axis and bisected by the axis.

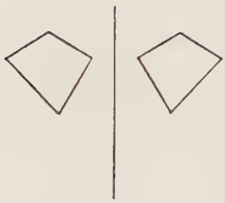

FIG. 32.

87. Definition. Two figures have an axis of symmetry when, with regard to this straight, every point of each has its symmetrical point on the other.

One figure has an axis of symmetry when, with regard to this straight, every point of the figure has its symmetrical point on the figure.

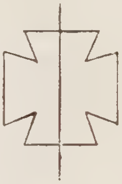

FIG. 33.

One figure is called symmetrical when it has an axis of symmetry.

Any figure has, with regard to any given straight as axis, always one, and only one, symmetrical figure.

88. Theorem. *An angle is symmetrical with regard to its bisector and the end-points of congruent sects from the vertex are symmetrical.*

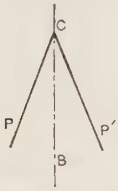

FIG. 34.

Proof. Their sect is bisected at right angles by the angle-bisector.

89. Definition. A symmetrical quadrilateral with a diagonal as axis is called a *deltoid.*

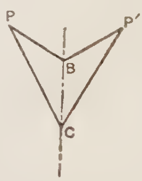

FIG. 35.

90. Definition. A sect whose end-points are the bisection-points of opposite sides of a quadrilateral is called a *median.* So is the sect from a vertex of a triangle to the bisection-point of the opposite side.

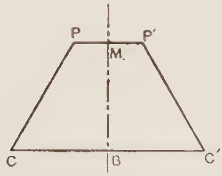

FIG. 36.

91. Definition. A symmetrical quadrilateral with a median as axis is called a *symtra.*

Ex. 36. In a r't $\triangle$ if to set off one acute $\measuredangle$, α, in the other, β, bisects, so is it with β's sides.

Ex. 37. The st' through the bisection-point of the base of a $\dotplus\triangle$, and the opposite vertex is $\perp$ to the base and bisects $\measuredangle$.

Ex. 38. The r't bi' of the base of a $\dotplus\triangle$ bisects $\measuredangle$ at the vertex.

Ex. 39. The $\perp$ from vertex bisects base and $\measuredangle$ in a $\dotplus\triangle$.

Ex. 40. The bisector of $\measuredangle$ at vertex of a $\dotplus\triangle$ is r't bi' of base.

Ex. 41. If a r't bi' of a side contains a vertex, the $\triangle$ is $\dotplus$.

Ex. 42. The bisector of an exterior $\measuredangle$ at vertex of $\dotplus\triangle$ is $\parallel$ to base, and inversely.

Ex. 43. The end of sect from intersection of congruent sides of a $\dotplus\triangle$ costraight and $\equiv$ to one determines with end of other a $\perp$ to base.

Ex. 44. To erect a $\perp$ at the end-point of a sect without producing the sect.

Ex. 45. A $\parallel$ to one side of an $\measuredangle$ makes with its bisector and other side a $\dotplus\triangle$.

Ex. 46. The bisectors of the $\equiv\measuredangle$s of a $\dotplus\triangle$ are $\equiv$.

Ex. 47. Every symmetrical quadrilateral not a deltoid is a symtra.

Ex. 48. The intersection point of two symmetrical straights is on the axis.

Ex. 49. The bisector of an angle is symmetrical to the bisector of the symmetrical angle.

Ex. 50. A figure made up of a straight and a point is symmetrical.

Ex. 51. In any deltoid [1] One diagonal (the axis) is the perpendicular bisector of the other. [2] One diagonal (the axis) bisects the angles at its two vertices. [3] Sides which meet on one diagonal (the axis) are con-

gruent; so each side equals one of its adjacent sides. [4] One diagonal (not the axis) joins the vertices of congruent angles and makes congruent angles with the congruent sides. [5] The triangles made by one diagonal (the axis) are congruent. [6] One diagonal (not the axis) makes two isosceles triangles.

Ex. 52. Any quadrilateral which has one of the six preceding pairs of properties (Ex. 51) is a deltoid.

Ex. 53. A quadrilateral with a diagonal which bisects the angle made by two sides, and is less than each of the other two sides, and these sides congruent, is a deltoid with this diagonal as axis.

Ex. 54. A quadrilateral with a side meeting a congruent side in a greater diagonal which is opposite congruent angles is a deltoid with that diagonal as axis.

Ex. 55. In any symtra [1] Two opposite sides are parallel, and have a common perpendicular bisector. [2] The other two sides are congruent and make congruent angles with the parallel sides.

[3] Each angle is congruent to one and supplemental to the other of the two not opposite it.

[4] The diagonals are congruent and their parts adjacent to the same parallel are congruent.

[5] One median bisects the angle between the two diagonals, and also the angle between the non-parallel sides (produced).

Ex. 56. Any quadrilateral which has one of the preceding five pairs of properties (Ex. 55) is a symtra.

92. Definition. A *trapezoid* is a quadrilateral with two sides parallel.

93. Definition. A *parallelogram* is a quadrilateral with each side parallel to another (its opposite).

94. Definition. A parallelogram with one angle right is called a *rectangle*. A parallelogram with two consecutive sides congruent is called a *rhombus*. A rectangle which is a rhombus is called a *square*.

95. Theorem. *The opposite sides and angles of a*

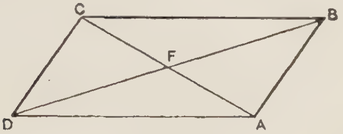

Fig. 37.

parallelogram are congruent, and its diagonals bisect each other.

Proof. $\triangle ABC \equiv \triangle ADC$ [side and 2 adjoining $\measuredangle$s$\equiv$]. $\therefore BC \equiv AD$. $\therefore$ (as in 82) $AF \equiv FC$ and $BF \equiv FD$.

96. Theorem. *If three parallels make congruent sects on one transversal, they do on every transversal.*

Given $a \parallel b \parallel c$, also $AB \equiv BC$.

To prove $FG \equiv GH$.

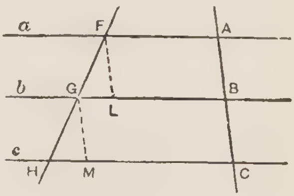

Fig. 38.

Proof. Take $FL \parallel GM \parallel AB$. Then $FL \equiv AB \equiv BC \equiv GM$ [95, opposite sides of a $\parallel$gm are$\equiv$]. $\therefore \triangle FLG \equiv \triangle GMH$ [side and 2 adjoining $\measuredangle$s$\equiv$]. $\therefore FG \equiv GH$.

97. Corollary to 96. A straight through the bisection-point of one side of a triangle and parallel to a second side bisects the third side. [In figure let F coincide with A.]

98. Inverse of 97. The straight through the bisection-points of any two sides of a triangle is parallel to the third side. [For, by 97, it is identical with the $\parallel$ to the third side through either bisection-point.]

99. Theorem. *The sect whose end-points are the bisection-points of two sides of a triangle is congruent to each sect made in bisecting the third side.*

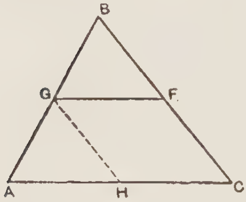

FIG. 39.

Proof. By 97 $GH \parallel BC$ bisects AC. Since (by 98) $FG \parallel CH$, $\therefore$ (by 95) $FG \equiv CH$.

100. Theorem. *If two sides of a quadrilateral are congruent and parallel it is a parallelogram.*

Given $AB \equiv$ and $\parallel CD$.

Proof. $\triangle ABC \equiv \triangle ADC$.

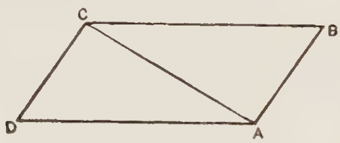

FIG. 40.

$$\therefore \measuredangle ACB \equiv \measuredangle CAD. \quad \therefore CB \parallel AD.$$

Ex. 57. Every straight through the intersection of its diagonals cuts any parallelogram into congruent trapezoids.

Ex. 58. A quadrilateral with each side equal to its opposite is a parallelogram.

Ex. 59. A quadrilateral with a pair of opposite sides equal, and each greater than a diagonal, making equal alternate angles with the other sides, is a parallelogram.

Ex. 60. A quadrilateral with a side equal to its opposite, and less than a diagonal opposite equal angles, is a parallelogram.

Ex. 61. A quadrilateral with each angle equal to its opposite is a parallelogram.

Ex. 62. A quadrilateral whose diagonals bisect each other is a parallelogram.

101. Theorem. In any sect AB there are always two, and only two, points, C, D, such that $AC \equiv CD \equiv DB$.

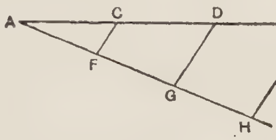

FIG. 41.

Proof. Take on any ray from A, any sect AF, and a sect $FG \equiv AF$, and a sect $GH \equiv FG$. Take $FC \parallel GD \parallel HB$. Then, by 96, $AC \equiv CD \equiv DB$.

Suppose two other such points C', D'. Then, by 98, $C'F \parallel D'G$. Now $HB' \parallel GD'$ (by 96) makes $D'B' \equiv D'C'$. $\therefore$ from our hypothesis and III 1, B' is identical with B. $\therefore$ since $GD \parallel HB$ (by IV) D' is identical with D. $\therefore$ since $FC \parallel GD$ (by IV) C' is identical with C.

102. The two points, C, D, of the sect AB such that $AC \equiv CD \equiv DB$ may be called the *trisection-points* of AB.

103. Theorem. *The three medians of a triangle are copunctal in that trisection-point of each remote from its vertex.*

Proof. Any median AG must meet any other CF, since A and G are on different sides of the straight CF, and so the cross of st' AG with st' CF is on sect AG, and similarly it is on sect CF. If P, Q, are bisection-points of OC and OA, then (by 98 and 99) $PQ \parallel$ and $\equiv GF$. $\therefore$ by 100 $PQFG$ is a $\parallel$gm and (by 95) PF and QG bisect each other.

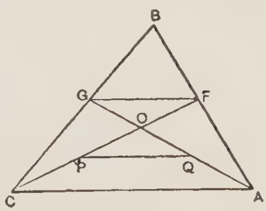

FIG. 42.

104. Definition. The cointersection-point of its medians is called the triangle's *centroid.*

105. Definition. A perpendicular from a vertex to the straight of the opposite side is called an *altitude* of the triangle. This opposite side is then called the *base.* The perpendicular from a vertex of a parallelogram to the straight of a side not through this vertex is called the altitude of the parallelogram with reference to this side as base.

Ex. 63. The bisectors of the four angles which two intersecting straights make with each other form two straights perpendicular to each other.

Ex. 64. If four coinitial rays make the first angle congruent to the third, and the second congruent to the fourth, they form two straights.

Ex. 65. How many congruent sects from a given point to a given straight?

Ex. 66. Does the bisector of an angle of a triangle bisect the opposite side?

Ex. 67. The bisectors of vertical angles are costraight.

Ex. 68. If two isosceles triangles be on the same base the straight determined by their vertices bisects the base at right angles.

Ex. 69. Suppose a $\triangle$ to be 3 bars freely jointed at the vertices. Is it rigid? Are the $\measuredangle$s fixed and the joints of no avail? Of what theorem is this a consequence? How is it with a jointed quadrilateral? Why?

Ex. 70. Joining the bisection-points of the sides of a $\triangle$ cuts it into $4 \equiv \triangle$s.

Ex. 71. Joining the bisection-points of the consecutive sides of a quadrilateral makes a $\parallel$ g'm.

Ex. 72. The medians of a quadrilateral and the sect joining the bisection-points of its diagonals are all three bisected by the same point.

Ex. 73. If the bisection-points of two opposite sides

of a $\parallel$ g'm are joined to the vertices the diagonals are trisected.

Ex. 74. The $\perp$s from any point in the base of an $\dotplus\triangle$ to the sides are together an altitude.

Ex. 75. The diagonals of a rectangle are $\cong$, of a rhombus are $\perp$.

Ex. 76. If 2 $\parallel$s are cut by a transversal, the bisectors of the interior $\measuredangle$s make a rectangle.

Ex. 77. The angle-bisectors of a rectangle make a square.

Ex. 78. If the $\measuredangle$s adjoining one of the $\parallel$ sides of a trapezoid are $\cong$, so are the others.

Ex. 79. The bisectors of the interior $\measuredangle$s of a trapezoid make a quad' with 2 r't $\measuredangle$s.

Ex. 80. The bisection-point of one sect between $\parallel$s bisects any through it.

Ex. 81. The $\cong$ altitudes in $\dotplus\triangle$ make with the base $\measuredangle$s $\cong$ to those made in bisecting the other $\measuredangle$.

Ex. 82. Through a given point within an $\measuredangle$ draw a sect terminated by the sides and bisected by the point.

Ex. 83. Sects from the vertex to the trisection-points of the base of $\dotplus\triangle$ are $\cong$.

Ex. 84. If the $\measuredangle$s made by producing a side of a $\triangle$ are $\cong$, so are the other sides.

Ex. 85. If a quad' has 2 pairs of congruent consecutive sides, the other $\measuredangle$s are $\cong$.

Ex. 86. Two $\triangle$s are $\cong$ if two sides and one's median are respectively $\cong$.

Ex. 87. Two $\dotplus\triangle$s are $\cong$ if one $\measuredangle$ and altitude are $\cong$ to the corresponding.

Ex. 88. Two $\triangle$s are $\cong$ if a side, its altitude and an adjoining $\measuredangle$ are respectively $\cong$.

Ex. 89. If 2 altitudes are $\cong$ the $\triangle$ is $\dotplus$.

Ex. 90. Two $\triangle$s are $\cong$ if two sides and one's altitude are $\cong$ to the corresponding.

Ex. 91. Two $\triangle$s are $\cong$ if a side, its altitude and median are respectively $\cong$.

Ex. 92. Two $\triangle$s are $\cong$ if a side and the other 2 altitudes are respectively $\cong$.

Ex. 93. Two $\triangle$s are $\cong$ if a side, an adjoining $\measuredangle$ and its bisector are respectively $\cong$.

Ex. 94. Two equilateral $\triangle$s are $\cong$ if an altitude is $\cong$..

Ex. 95. The bisector is within the $\measuredangle$ made by altitude and median.

Ex. 96. In a right $\triangle$ one bisector also bisects $\measuredangle$ between its altitude and median.

Ex. 97. Two sects from the vertices of a $\triangle$ to the opposite sides cannot bisect each other.

Ex. 98. The $\perp$s from 2 vertices of a $\triangle$ upon the median from the third are $\cong$.

Ex. 99. Two $\parallel$ g'ms having an $\measuredangle$ and the including sides $\cong$ are $\cong$.

Ex. 100. The $\perp$ from the circumcenter to a side is half the sect from the opposite vertex to the orthocenter.

Ex. 101. The centroid is the trisection point of the sect from orthocenter to circumcenter remote from the orthocenter

CHAPTER V.

THE CIRCLE.

106. Definition. If C is any point in a plane α, then the aggregate of all points A in α, for which the sects CA are congruent to one another, is called a *circle*. [$\odot C(CA)$.]

C is called the *center of the circle*, and CA the *radius*.

107. Theorem. Any ray from the center of a circle and in its plane α cuts the circle in one, and only one, point.

108. Theorem. Any straight through its center and in its plane α cuts the circle in two, and only two, points, and these are on opposite sides of its center.

Proof. On each of the two rays determined in this straight by the center there is (by III 1) one, and only one, sect congruent to the radius of the circle.

109. Definition. A sect whose end-points are on the circle is called a *chord*.

110. Definition. Any chord through the center is called a *diameter*.

111. Theorem. Every diameter is bisected by the center of the circle.

112. Theorem. No circle can have more than one center.

Proof. If it had two, the diameter through them would have two bisection-points, which (by 82) is impossible.

113. Theorem. *The straight through the bisection-point of a chord, and the center of the circle, is perpendicular to the chord.*

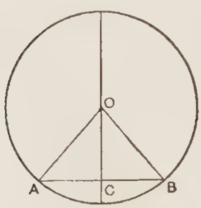

FIG. 43.

Proof. $\triangle ACO \equiv \triangle BCO$, [$\triangle$s with 3 sides $\equiv$ are $\equiv$]; $\therefore \measuredangle ACO \equiv \measuredangle BCO$. But they are adjacent; $\therefore$ by definition $CO \perp$ to AB.

114. Corollary to 113. The circle is symmetrical with regard to any one of its diameters as axis.

115. Corollary to 113. If with the end-points of a sect each of two points gives congruent sects the two determine its perpendicular bisector.

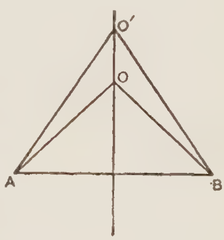

FIG. 44.

116. Corollary to 115. If two circles have two points in common their center-straight is the perpendicular bisector of their common chord.

117. Theorem. *The perpendicular bisecting any chord contains the center. The perpendicular from the center to a chord bisects it.*

Proof. By 113 the three properties pertain to one straight. But any two suffice to determine that straight.

118. Corollary to 117. Every point which taken

with two points gives congruent sects is on the perpendicular bisector of their sect.

119. Theorem. *Every point on the perpendicular bisector of a sect taken with its end-points gives congruent sects.*

120. Theorem. A straight cannot have more than two points in common with a circle.

Proof. If it had a third, then, since (by 117) the perpendicular bisecting any chord contains the center, there would be two perpendiculars from the center to the same straight, which (by 47) is impossible.

121. Theorem. Chords which mutually bisect are diameters.

Proof. The perpendicular bisector of each contains the center.

122. Theorem. *Circles with three points in common are identical.*

Proof. The center is on the perpendicular bisectors of the chords.

123. Theorem. *Any three points not costraight determine a circle.*

Proof. If A, B, C be not costraight, bisect (by 82) AB at D and BC at F by perpendiculars. [Take (by III 4) angles $\equiv$ to $\measuredangle C$ in 84.] These perpendiculars (by 77) meet, say at O. Therefore (by 119) $AO \equiv BO \equiv CO$. Therefore A, B, C are on the circle with center O and radius AO. By 122 the three are on no other circle.

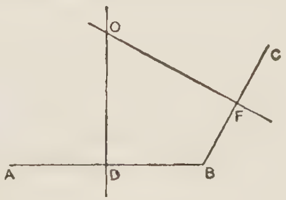

Fig. 45.

124. Corollary to 123 and 120. [Points on the same circle are called *concyclic.*] Every three points are costraight or concyclic. No three points are costraight and concyclic.

125. Definition. The circle through the vertices of a triangle is called its *circumcircle*, $\odot O(R)$, and the center O of the circumcircle is called the *circumcenter* of the triangle; its radius, R, the *circum-radius.*

126. Corollary to 123. The three perpendicular bisectors of the sides of a triangle are copunctal in its circumcenter.

127. Theorem. *The three altitudes of a triangle are copunctal.*

Given the $\triangle ABC$. To prove that the straights through A, B, C perpendicular to the straights a, b, c respectively, are copunctal.

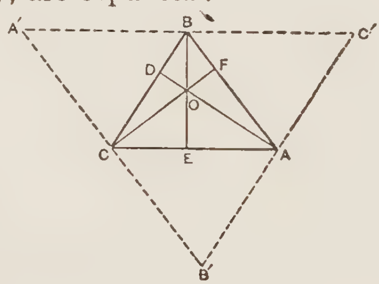

FIG. 46.

Proof. By 66, through A, B, C take $B'C'$, $A'C'$, $A'B' \parallel BC$, AC, AB respectively. $\therefore \triangle AB'C \equiv \triangle ABC \equiv \triangle ABC'$ [$\triangle$s with a side and 2 adjoining $\measuredangle$s$\equiv$ are $\equiv$]. $\because AB' \equiv AC'$, and AD is the $\perp$bi' of $B'C'$ [$\perp$ to 1st of 2 $\parallel$s is $\perp$ to 2nd].

Similarly $BE \perp$bi' of $A'C'$; and $CF \perp$bi' of $A'B'$. $\therefore AD$, BE, CF are copunctal by 126.

128. Definition. The point of cointersection of the three altitudes is called the *orthocenter* of the triangle.

129. Theorem. If any ray, l, be taken within a given angle, $\measuredangle(h, k)$, the bisectors m, n of the two angles so made form an angle congruent to each of the angles made in bisecting the given angle.

Proof. On the other side of h from k take $\measuredangle(h, k') \equiv \measuredangle(k, n)$. Then since $\measuredangle(h, n) \equiv \measuredangle(n, h)$ and $\measuredangle(n, k) \equiv \measuredangle(h, k')$, $\therefore$ (by 49) $\measuredangle(h, k) \equiv \measuredangle(n, k')$.

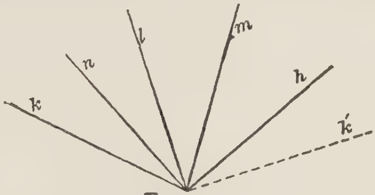

FIG. 47.

But since $\measuredangle(n, l) \equiv \measuredangle(k, n)$, $\therefore$ $\measuredangle(n, l) \equiv \measuredangle(k', h)$. But also $\measuredangle(l, m) \equiv \measuredangle(h, m)$. $\therefore$ (by 49) $\measuredangle(n, m) \equiv \measuredangle(k', m)$, $\therefore$ m bisects $\measuredangle(n, k')$. $\therefore$ (by 48) $\measuredangle(m, n) \equiv \measuredangle(b, h)$ where b bisects $\measuredangle(h, k)$.

130. Theorem. The bisectors of adjacent angles make a right angle.

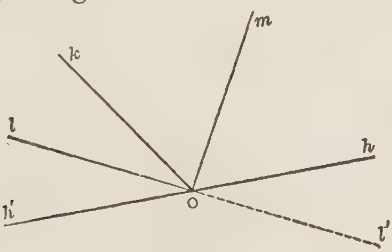

FIG. 48.

Proof. Extend one of the bisectors, as l, through the vertex O. Then $\measuredangle(l', h) \equiv \measuredangle(l, h')$ [vertical

$\measuredangle$s are $\equiv$]. $\therefore$ (by III 5) $\measuredangle(l', h) \equiv \measuredangle(l, k)$. But by hypothesis, $\measuredangle(h, m) \equiv \measuredangle(k, m)$. $\therefore$ (by 49) $\measuredangle(l, m) \equiv \measuredangle l', m)$. $\therefore$ by definition $\measuredangle(l, m)$ is right.

Ex. 102. If a straight satisfy any two of the following conditions it also satisfies the others:

1. Passing through the center.
2. Perpendicular to the chord.
3. Bisecting the chord.
4. Bisecting the angle at the center.

Ex. 103. Every axis of symmetry for a circle contains the center.

Ex. 104. Where are the bisection-points of a set of parallel chords?

Ex. 105. Where are the bisection-points of a set of equal chords?

Ex. 106. If from any point three sects drawn to a circle are congruent, that point is the center.

131. Theorem. If any ray l be taken without a given angle, $\measuredangle(h, k)$, and of the two angles so formed, one, $\measuredangle(k, l)$, be within the other, $\measuredangle(h, l)$, then the angle formed by their bisectors, $\measuredangle(b, n)$, is congruent to each of the angles as $\measuredangle(k, m)$ made in bisecting the given angle.

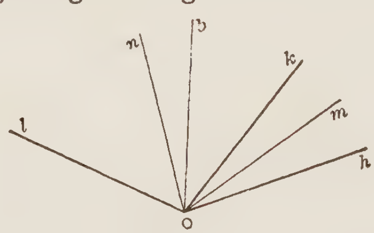

Fig. 49.

Proof. By 129, since k is within $\measuredangle(l, h)$, $\therefore$ $\measuredangle(m, n) \equiv \measuredangle(b, l)$. But by hypothesis $\measuredangle(n, k) \equiv \measuredangle(l, n)$, $\therefore$ (by 49) $\measuredangle(k, m) \equiv \measuredangle(n, b)$.

132. Definition. An angle whose vertex is on a circle of which its sides contain chords is called an *inscribed angle*, and said to be *upon* the chord between its sides.

133. Theorem. *Inscribed angles upon the same chord* and the same side of it *are congruent.*

Proof. 1st. If the chord BC be a diameter the straight through the vertex A of any inscribed angle and the center O makes two isosceles triangles. $\therefore$ bisecting $\sphericalangle BOF$ we get $\sphericalangle DOF \equiv \sphericalangle BAO$. In same way $\sphericalangle HOF \equiv \sphericalangle CAO$. $\therefore$ (by 49) $\sphericalangle BAC \equiv \sphericalangle DOH$, which (by 130) is right.

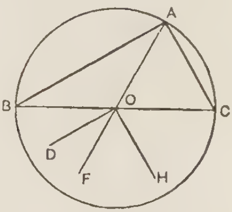

FIG. 50.

2d. If the vertex A be on the same side of the straight BC as the center O, then sect OA cannot cut BC, and (by III 1) the center O is between A and the other point A' of the circle on the straight AO. If now ray OA be costraight with a side of $\sphericalangle BOC$, then $\triangle AOC$ being isosceles, the bisector OF of $\sphericalangle BOC$ makes $\sphericalangle FOC \equiv \sphericalangle BAC$.

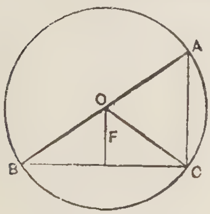

FIG. 51.

Again if ray OA' is within $\sphericalangle BOC$, then the bisectors OF and OH make

$\sphericalangle A'OF \equiv \sphericalangle OAB$ and
$\sphericalangle A'OH \equiv \sphericalangle OAC$. $\therefore$ (by 49)
$\sphericalangle BAC \equiv \sphericalangle FOH$, which

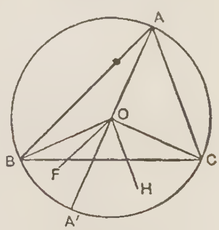

FIG. 52.

(by 129) is congruent to each angle made in bisecting BOC.

If, however, ray OA' is without $\measuredangle BOC$, then the bisector OD of $\measuredangle A'OB$ makes $\measuredangle A'OD \equiv \measuredangle OAB$, and the bisector OE of $\measuredangle A'OC$ makes $\measuredangle A'OE \equiv \measuredangle OAC$. $\therefore$ (by 49) $\measuredangle BAC \equiv \measuredangle DOE$, which (by 131) is congruent to each angle made in bisecting BOC.

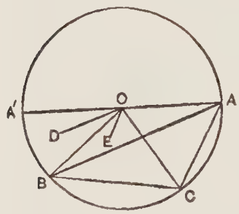

FIG. 53.

3d. If the vertex A and the center O be on opposite sides of the straight BC. Let A' be the other point of the circle on the straight AO. Then the six angles of the two triangles ABC, $A'BC$ together form four right angles. But by case 1st, the two angles at C form a right angle, likewise the two at B. $\therefore$ $\measuredangle BAC$ is the supplement of $\measuredangle BA'C$.

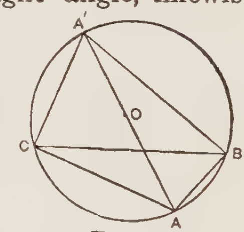

FIG. 54.

134. Corollary to 133. The inscribed angle upon a diameter is right.

135. Definition. A polygon whose sides are congruent and whose angles are congruent is called *regular*.

136. Definition. A polygon whose vertices are concyclic is called *cyclic*.

137. Corollary to 133. In a cyclic quadrilateral the opposite angles are supplemental.

138. Theorem. *If a straight have one point in common with a circle and be not perpendicular to the radius to that point, it has also a second point in common with the circle.*

Let the straight a have the point P in common with the $\odot C[CP]$ and be not $\perp$ to CP.

From C drop $CM \perp$ to a. From M on a set off $MP' \equiv MP$.

$\therefore CP' \equiv CP$. $\therefore P'$ is on $\odot C[CP]$.

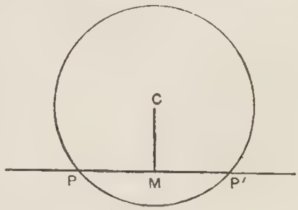

FIG. 55.

139. Definition. A straight which has two points in common with a circle is called a *secant.*

140. Theorem. *A straight perpendicular to a diameter at an end-point has only this end-point in common with the circle.*

Proof. Any chord is (by 117) bisected by the perpendicular from the center.

141. Definition. A straight which has only one point in common with a circle is called a *tangent* to the circle, and the point is called the *point of contact.*

142. Theorem. If BC be perpendicular to AB, and D any point on the straight AB other than B, and on ray CD we take sect $CF \equiv CB$, then F is within sect CD.

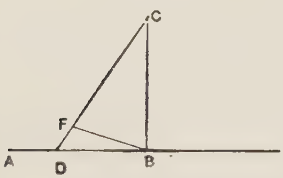

FIG. 56.

Proof. Otherwise since $\triangle CBF$ is isosceles, two angles of a triangle would each be right or each obtuse, which (by 79) is impossible.

143. Theorem. If the rays of one angle are within another the angles are not congruent.

Proof. For suppose $\sphericalangle(h, m) \equiv \sphericalangle(k, l)$ and k, l within $\sphericalangle(h, m)$. On the other side of m from the points of h there is a ray n such that $\sphericalangle(m, n) \equiv \sphericalangle(h, k)$. $\therefore$ (by 49) $\sphericalangle(k, n) \equiv \sphericalangle(h, m)$. $\therefore$ from our hypothesis $\sphericalangle(k, n) \equiv \sphericalangle(k, l)$, which (by III 4) is impossible.

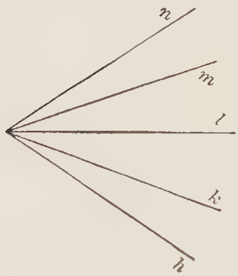

FIG. 57.

144. Theorem. If P be a point within the triangle ABC, then angle APC is not congruent to angle ABC.

Proof. The ray BP must (by 30) have on it a point D within the sect AC. $\therefore PD$ is within $\sphericalangle APC$. From P take $PF \parallel BC$. It makes $\sphericalangle FPD \equiv \sphericalangle CBD$. From P take $PG \parallel AB$. It makes $\sphericalangle GPD \equiv \sphericalangle ABD$. $\therefore$ (by 49) $\sphericalangle FPG \equiv \sphericalangle CBA$.

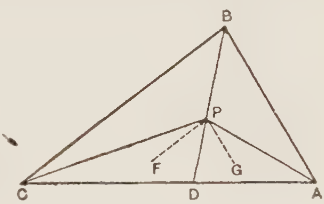

FIG. 58.

If now we supposed $\sphericalangle CBA \equiv \sphericalangle CPA$ we should have $\sphericalangle FPG \equiv \sphericalangle CPA$, which (by 143) is impossible.

145. Theorem. *If two triangles have a side in common and the angles opposite it congruent, and with vertices on the same side of it, the four vertices are concyclic.*

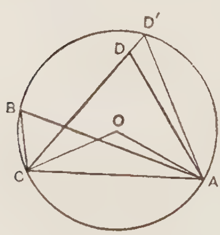

FIG. 59.

Proof. If the circle through A, B, C did not contain D, then (by 138) it would have a second point in common with the

straight AD or CD, or else we would have $OC \perp CD$ and $OA \perp AD$.

But in this last case the whole circle except points A and C would be within $\measuredangle ADC$. For the center O would be within $\measuredangle ADC$, being then on the bisector of $\measuredangle ADC$ since, $\triangle ADC$ being isosceles, $\triangle AOD \equiv \triangle COD$ [3 sides $\equiv$]. Hence the point B would be within $\triangle ADC$, which (by 144) is impossible. But it is just as impossible that AD or CD (besides A or C) should have a point D' on the circle other than D. For then we would have $\measuredangle ADC \equiv \measuredangle AD'C$, which (by 79) is impossible.

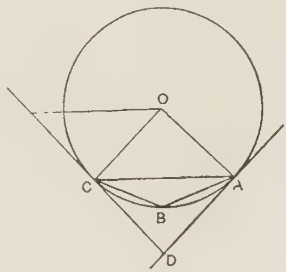

FIG. 60.

146. Theorem. *If two opposite angles of a quadrilateral are supplemental it is cyclic.*

Proof. Given the $\measuredangle CDA$ is the supplement of $\measuredangle B$. On the circle determined by A, B, C take a point D' on the same side of AC as D. Then $\measuredangle D' \equiv \measuredangle D$, being each the supplement of $\measuredangle B$. $\therefore$ (by 145) D is concylic with ACD', that is, with ABC.

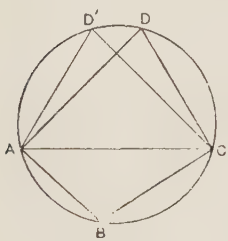

FIG. 61.

147. Corollary to 146. A quadrilateral is cyclic if an angle is congruent to the angle adjacent to its opposite.

Ex. 107. Defining a *tanchord* angle as one between a tangent to a circle and a chord from the point of contact, prove it congruent to an inscribed angle on this chord.

Ex. 108. An angle made by two chords is how related to the angles at the center on chords joining the end-points of the given chords?

Ex. 109. The vertices of all right-angled triangles on the same hypothenuse are concyclic.

Ex. 110. Tangents to a circle from the same external point are congruent, and make congruent angles with the straight through that point and the center.

Ex. 111. Two congruent coinitial chords are symmetric with respect to the coinitial diameter.

Ex. 112. If triangles on the same base and on the same side of it have the angles opposite it equal, the bisectors of these angles are copunctal.

Ex. 113. The end-points of two congruent chords of a circle are the vertices of a symmetrical trapezoid.

Ex. 114. The chord which joins the points of contact of parallel tangents to a circle is a diameter.

Ex. 115. A parallelogram inscribed in a circle must have diameters for diagonals.

Ex. 116. Of the vertices of a triangle and its orthocenter, each is the orthocenter of the other three.

Ex. 117. At every point on the circle can be taken one, and only one, tangent, namely, the perpendicular to the radius at the point.

Ex. 118. The perpendicular to a tangent from the center of the circle cuts it in the point of contact.

Ex. 119. The perpendicular to a tangent at the point of contact contains the center.

Ex. 120. The radius to the point of contact is perpendicular to the tangent.

Ex. 121. An inscribed ||g'm is a rectangle.

Ex. 122. The bisector of any ∡ of an inscribed quad' intersects the bisector of the opposite exterior ∡ on the ⊙.

Ex. 123. The ⊙ with one of the ≡ sides of a ∔ △ as diameter bisects the base.

Ex. 124. The radius is $=$ to the side of a regular inscribed hexagon.

Ex. 125. If the opposite sides of an inscribed quad' be produced to meet, the bisectors of the ∡s so formed are $\perp$.

Ex. 126. The circles on 2 sides of a △ as diameters intersect on the third side (in the foot of its altitude).

Ex. 127. The altitudes of a △ are the ∡ bisectors of its *pedal* △ (the feet of the altitudes).

Ex. 128. AB a diameter; AC any chord; CD tangent; $BD \perp CD$, meets AC on $\odot B(BA)$.

Ex. 129. Find a point from which the three rays to three given points make $=$ ∡s.

Ex. 130. The circum⊙s of 3 △s made by 3 points on the sides of a △, 2 with their vertex, are copunctal.

V. The Archimedes Assumption.*

V. Let A_1 be any point on a straight between any given points A and B; take then the points A_2, A_3, A_4, . . . , such that A_1 lies between A and A_2, furthermore A_2 between A_1 and A_3, further A_3 between A_2 and A_4, and so on, and also such that the sects AA_1, A_1A_2, A_2A_3, A_3A_4, . . . , are congruent; then in the series of points A_2, A_3, A_4, . . . , there is always such a point A_n, that B lies between A and A_n.

148. This postulate makes possible the introduction of the continuity idea into geometry. We have not used it, and will not, since the whole of the ordinary school-geometry can be constructed with only Assumptions I–IV.

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* Archimedis Opera, rec. Heiberg, vol. I, 1880, p. 11.

CHAPTER VI.

PROBLEMS OF CONSTRUCTION.

Existence theorems on the basis of assumptions I–V, and the visual representation of such theorems by graphic constructions.

Graphic solutions of the geometric problems by means of ruler and sect-carrier.

149. Convention. What are called *problems of construction* have a double import. Theoretically they are really theorems declaring that the existence of certain points, sects, straights, angles, circles, etc., follows logically by rigorous deduction from the existences postulated in our assumptions. Thus the possibility of solving such problems by elementary geometry is a matter absolutely essential in the logical sequence of our theorems.

So, for example, we have shown (in 101) that a sect has always trisection points, and this may be expressed by saying we have solved the problem to trisect a sect. Now it happens that a solution of the problem to trisect any angle is impossible with only our assumptions. Thus any reference to results following from the trisection of the angle would be equivalent to the introduction of additional assumptions.

But problems of construction, on the other hand, may have a reference to practical operations, usually for drawing on a plane a picture which shall serve as an approximate graphic representation of the data and results of the existential theorem.

Our Assumptions I postulate the existence of a straight as the result of the existence of two points. This may be taken as authorizing the graphic designation of given points and the graphic operation to join two designated points by a straight, and as guaranteeing that this operation can always be effected. Confining ourselves to plane geometry, on the basis of the same Assumptions I, we authorize the graphic operation to find the intersection-point of two coplanar non-parallel straights, and guarantee that this may always be accomplished.

To practically perform these graphic operations, that is for the actual drawing of pictures which shall represent straights with their intersections, we grant the use of a physical instrument whose edge is by hypothesis straight, namely, the straight-edge or ruler.

Thus Assumptions I give us as assumed constructions, or as solved, the fundamental problem of plane geometry:

Problem 1. (*a*) *To designate a given point of the plane;* (*b*) *to draw the straight determined by two points; to find the intersection of two non-parallel straights*

150. Our Assumptions III postulate the existence on any given straight from any given point of it toward a given side, of a sect congruent to a given sect.

This may be taken as authorizing and guaranteeing the graphic operation involved in what may be called

Problem 2. *To set off a given sect on a given straight from a given point toward a given side.*

A physical instrument for actual performance of this construction in drawing might be called a sect-carrier. Our straight-edge will also serve as sect-carrier if we presume that the given sect may be marked off on it, and it then made to coincide with the given straight with one of the marked points in coincidence with the given point of the straight.

Notice that in these graphic interpretations we freely use the terminology of motion, while the real existential theorems themselves are independent of motion, underlie motion, and explain motion. We assume that the motion of our physical instruments is rigid.

151. We now announce the important theorem that in our geometry all graphic problems can be solved, all graphic constructions effected, merely by using problems 1 and 2.

Theorem. *Those geometric construction problems (existential theorems) solvable by employing exclusively Assumptions I–V are necessarily graphically solvable by means of ruler and sect-carrier.*

The demonstration will consist in solving with problems 1 and 2 the three following problems:

152. Problem 3. *Through a given point to draw a parallel to a given straight.*

Given the straight AB and the point P.

Construction. Join P with any point A of AB by Prob. 1. On the straight PA beyond A take (by Prob. 2) $AC \equiv AP$. Join C with any other point B of AB. On the straight CB beyond B take $BQ \equiv BC$. PQ is the parallel sought.

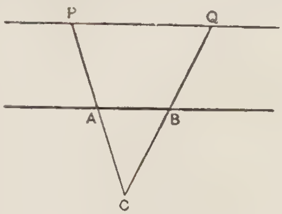

FIG. 62.

Proof. By 98.

153. Problem 4. *To draw a perpendicular to a given straight.*

Construction. Let A be any point of the given straight. Set off from A on this straight toward both sides two congruent sects, AB and AC, and then determine on any two other straights through A the points E and D, on the same side of AB, and such that

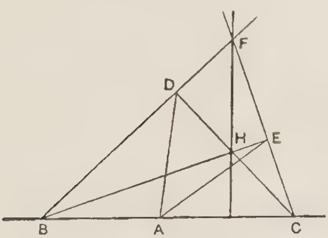

FIG. 63.

$AB \equiv AD \equiv AE$. Since $\measuredangle ABD$ and $\measuredangle ACE$ are angles at the bases of isosceles triangles, $\therefore$ they are acute, $\therefore$ the straights BD and CE meet in F, and also the straights BE and CD in H. Then FH is the perpendicular sought.

Proof. $\measuredangle BDC$ and $\measuredangle BEC$, as inscribed angles on the diameter BC, are (by 134) right. Since (by 127) the altitudes of $\triangle BCF$ are copunctal, $\therefore$ FH is $\perp$ to BC.

154. Problem 5. *To set off a given angle against a given straight*, or to construct a straight cutting a given straight under a given angle.

Given β the angle to be set off, and A its vertex.

Construction. We draw, by Prob. 3, the straight l through A $\parallel$ to the given straight against which the given angle β is to be set off. By Prob. 4 draw a straight $\perp$ to l and a straight $\perp$ to one side of β. Through any point B of the other side of β draw, by Prob. 3, $\parallel$s to these $\perp$s. Call their feet C and D. Then (by Probs. 4 and 3) draw from A a st' $\perp$ to CD. Call its foot E.

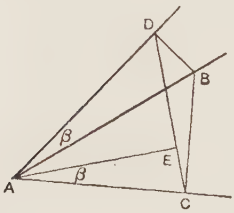

FIG 64.

Then $\measuredangle CAE \equiv \beta$. So EA will cut the given straight $\parallel$ to l under the given $\measuredangle \beta$.

Proof. Since $\measuredangle ACB$ and $\measuredangle ADB$ are right, so (by 146) the four points A, B, C, D are concyclic. Consequently $\measuredangle ACD \equiv \measuredangle ABD$ (by 133) being inscribed angles on the same chord AD and on the same side of it. Therefore their complements $\measuredangle CAE \equiv \measuredangle BAD$.

155. This completely demonstrates our theorem, 151, since the existential theorems in Assumptions II guarantee the solution of problems requiring no new graphic operations, such as to find a point within and a point without a given sect, and certain other problems of arrangement; while Assumption V would simply guarantee the finding of a point without a given sect by repeating a certain specific application of our Prob. 2.

156. In our geometry, though constantly using

graphic figures, we must never rely or depend upon them for any part of our proof. We must always take care that the operations undertaken on a figure also retain a purely logical validity.

157. This cannot be sufficiently stressed. In the right use of figures lies a chief difficulty of our investigation.

The graphic figure is only an approximate suggestive representation of the data. We cannot rely upon what we suppose to be our immediate perception of the relations in even the most accurate obtainable figure.

In rigorous demonstration, the figure can be only a symbol of the conceptual content covered by its underlying assumptions.

The logical coherence should not be dependent upon anything supposed to be gotten merely from perception of the figure. No statement or step can rest simply on what appears to be so in a figure. Every statement or step must be based upon an assumption, a definition, a convention, or a preceding theorem.

Yet the aid from figures, from sensuous intuition, is so inexpressibly precious, that any attempt even to minimize it would be a mistake. That treatment of a proposition is best which connects it most closely with a visualization of the figure, while yet not using, as if given by the figure, concepts not contained in the postulates and preceding propositions.

158. As an immediate result of Prob. 5, the proofs in Chapters I–V of our existential theorems give ruler

and sect-carrier solutions of the corresponding problems. We will now give some alternative solutions.

159. Problem 6. *At a given point A to make a right angle.*

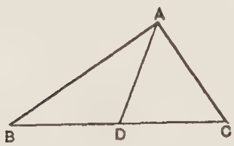

FIG. 65.

Solution. Draw through A any straight AD, and through D any other straight BC, and make $AD \equiv BD \equiv CD$. Then is $\measuredangle BAC$ right. [Inscribed angle on a diameter.]

160. Problem 7. *From a given point A to drop a perpendicular upon a given straight BC.*

Solution. By Prob. 6, at A construct a rt. $\measuredangle BAC$. Make $BD \equiv BA$. Draw $DE \parallel AC$. Make $BF \equiv BE$. Then is $AF \perp BC$.

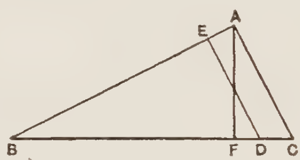

FIG. 66.

Proof. $\triangle ABF \equiv \triangle DBE$. [2 sides and inc. $\measuredangle \equiv$.]

161. Problem 8. *At any point A on a straight BC to erect the perpendicular.*

Solution. By Prob. 7, from any point without the straight drop to it a perpendicular. By Prob. 3, draw a parallel to this through A.

162. Problem 9. *To bisect a given sect AB.*

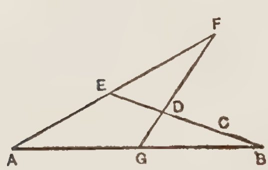

FIG. 67.

Construction. Draw through B any other straight BC. Make on it $BC \equiv CD \equiv DE$. Produce $AE \equiv EF$. Draw FDG. Then is $AG \equiv GB$.

Proof. D is the centroid of $\triangle ABF$.

163. Problem 5. *At a given point in a given straight to make an angle congruent to a given angle.*

Required, against the given ray AB of a, and toward a given side of a, the C-side, to make an angle $\equiv \measuredangle D$ (given).

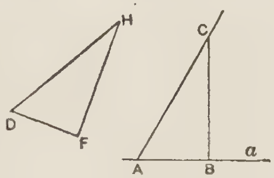

Fig. 68.

Construction. To one side of the given acute angle erect (by 161) $FH \perp DF$, meeting the other side at H. Take $AB \equiv DF$ and $BC \perp AB$ and $BC \equiv FH$. $\therefore$ (by III 6) $\measuredangle BAC \equiv \measuredangle FDH$.

164. Problem 10. *To bisect a given angle.*

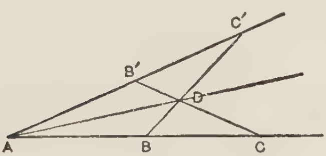

Fig. 69.

Construction. On one side of the given $\measuredangle A$ take any two points B, C. On the other side take $AB' \equiv AB$, and $AC' \equiv AC$. The sects BC' and $B'C$ intersect, say at D. AD is the desired bisector.

165. Problem. *To join two points by an arc containing a given angle.*

Let A, B be the two points, α the given angle. Make an angle BAC supplemental to α. Erect the perpendicular to AC at A, and to AB at the bisection-point. Their point of intersection is the center of the required circle. $\measuredangle AFB = \alpha$.

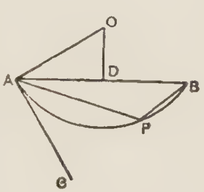

Fig. 70.

Proof. Their supplements

$$\measuredangle AOD = \measuredangle BAC \text{ (complements of } \measuredangle OAD).$$

166. Problem. *To describe a circle touching three given intersecting but not copunctal straights.*

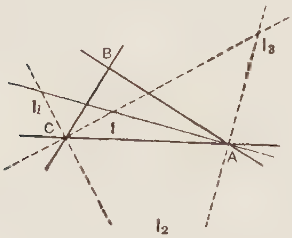

FIG. 71.

Construction. At the points of intersection draw the angle-bisectors. From the cross of any two of these bisectors, the perpendicular upon either of the three straights is the radius of a circle touching all three.

167. Definition. The cointersection-point of the three bisectors of the internal angles of a triangle, I, is called the triangle's *in-center* [r, the *in-radius*]; $\odot I(r)$ the *in-circle*.

168. Definition. A circle touching one side of a triangle and the other two sides produced is called an *escribed* circle, or *ex-*$\odot$. The three centers I_1, I_2, I_3 of the escribed circles $\odot I_1(r_1)$, $\odot I_2(r_2)$, $\odot I_3(r_3)$ of a triangle are called its *ex-centers*.

Ex. 131. A right angle can be trisected.

Ex. 132. To construct a triangle, given two sides and the included angle.

Ex. 133. To construct a triangle, given two angles and the included side.

Ex. 134. To construct a triangle, given two angles and a side opposite one of them.

Ex. 135. To describe a parallelogram, given two sides and the included angle.

Ex. 136. To construct an isosceles triangle, having given the base and the angle at the vertex.

Ex. 137. To erect a perpendicular to a sect at its end-point, without producing the sect or using parallels.

Hint. At this end-point against the given sect make

any acute angle. At any other point of the sect make toward this a congruent angle.

Beyond the intersection-point of the rays, make on this second ray a sect congruent to a side of this isosceles triangle. Its end is a point of the required perpendicular.

Ex. 138. Construct a circle containing two given points with center on a given straight.

Ex. 139. To draw an angle-bisector without using the vertex.

Ex. 140. Through a given point to draw a straight which shall make congruent angles with two given straights.

Ex. 141. In a straight find a point with which two given points give equal sects.

Ex. 142. From two given points on the same side of a straight to draw two straights intersecting on it and making congruent angles with it.

Ex. 143. To draw a straight through a given point between two given straights such that they intercept on it a sect bisected by the given point.

Ex. 144. Through a given point to draw a st' making $\equiv \measuredangle$s with the sides of a given $\measuredangle$.

Ex. 145. Construct $+ \triangle$ from b and h_b; from α and b; from β and $a+b$; from β and h_b; from b and β; from p and h_b; from p and α; from b and r.

Ex. 146. Construct r't $\triangle$ from α and h_c; from α and c; from c and r; from a and r; from β and r; from α and $a+b$; from R and r.

Ex. 147. Construct $\triangle$ from p, α, and h_a; from p, α, and β; from its pedal; from b, $a+c$, α; from α, h_b, p; from I_1, I_2, I_3.

Ex. 148. Without prolonging two sects, to find the bisector of the $\measuredangle$ they would make.

Ex. 149. Describe $\odot$ through two given points with center on given st'; with given radius.

Ex. 150. From one end of the hypothenuse lay off a sect on it congruent to the $\perp$ from the end of this sect to the other side.

Ex. 151. From + △ cut a trapezoid with 3 sides ≡.

Ex. 152. To inscribe a sq. in a given r't + △.

Ex. 153. Find point in side of + △ where ⊥ erected and produced to other side is ≡ to base.

Ex. 154. To describe a ⊙ which shall pass through a given point and touch a given st' at a given point.

Ex. 155. AB, AC, BD, CE, are chords. $BD \parallel AC$, $CE \parallel AB$. Then $AF \parallel DE$ is a tangent.

Ex. 156. To describe a ⊙ whose center shall be in one ⊥ side of a r't △ while the ⊙ goes through the vertex of the r't ∡ and touches the hypothenuse.

Ex. 157. To describe a ⊙ of given radius with center in one side of a given ∡ and tangent to the other side.

Ex. 158. Construct △ from a, α, and that t_α trisects a; from a and orthocenter; from a and centroid.

Ex. 159. Construct △ from α, β, R; from feet of medians; of altitudes.

Ex. 160. (Brahmagupta.) If the diagonals of an inscribed quad' are ⊥, the st' through their intersection ⊥ to any side bisects the opposite side.

CHAPTER VII.

SIDES, ANGLES, AND ARCS.

169. Convention. When a sect congruent to CD is taken on sect AB from A and its second end-point falls between A and B, then AB is said to be *greater* than CD; $(AB > CD)$. When an angle congruent to $\measuredangle(h, k)$ is set off from vertex O against one of the rays of $\measuredangle AOB$ toward the other ray, if its second side falls within $\measuredangle AOB$, then $\measuredangle AOB$ is said to be greater than $\measuredangle(h, k)$. In symbols, $\measuredangle AOB > \measuredangle(h, k)$.

170. Theorem. *If the first side of a triangle be greater than a second, then the angle opposite the first must be greater than the angle opposite the second.*

Given $BA > BC$.

To prove $\measuredangle C > \measuredangle A$.

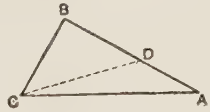

FIG. 72.

Proof. From B toward A take $BD \equiv BC$. The end-point D of this sect then, because $BA > BC$, is between A and B, that is within $\measuredangle ACB$, as is therefore also CD. Then is $\triangle BDC$ isosceles, $\therefore \measuredangle CDB \equiv \measuredangle DCB$. $\therefore \measuredangle ACB > \measuredangle BCD$ or $\measuredangle CDB$. But (by 79) $\measuredangle CDB > \measuredangle A$.

171. Inverse. If $\measuredangle A > \measuredangle B$, $\therefore a > b$.

Proof. [From 57 and 170.]

172. Definition. Except the perpendicular, any sect from a point to a straight is called an *oblique*.

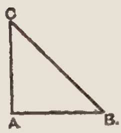

FIG. 73.

173. Theorem. *From a point to a straight any oblique is greater than the perpendicular.*

Proof. Since $\measuredangle CAB$ is r't, $\therefore$ (by 79) $\measuredangle A > \measuredangle B$. $\therefore$ (by 171) $a > b$.

174. Theorem. *Any two sides of a triangle are together greater than the third side.*

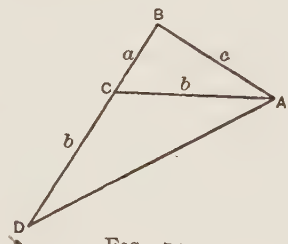

FIG. 74.

Proof. On st' BC, beyond C, take $CD \equiv CA$. $\therefore$ (by 57) $\measuredangle CDA \equiv \measuredangle CAD$. But AC is within $\measuredangle DAB$, $\therefore$ $\measuredangle DAB > \measuredangle DAC \equiv \measuredangle D$. $\therefore$ (by 171) $BD > AB$.

175. Theorem. (The ambiguous case.) *If two triangles have two sides of the one congruent respectively to two sides of the other, and the angles opposite one pair of congruent sides congruent, then the angles opposite the other pair are either congruent or supplemental.*

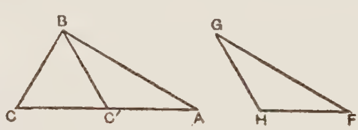

FIG. 75.

Hypothesis. $\triangle ABC$ and $\triangle FGH$ with $\measuredangle A \equiv \measuredangle F$, $AB \equiv FG$, and $BC \equiv GH$.

Conclusion. $\measuredangle C \equiv \measuredangle H$, or $\measuredangle C$ supplement of $\measuredangle H$.

Proof. At B against BA take, on the side toward C, the $\measuredangle ABC' \equiv \measuredangle G$. If ray BC' falls on ray BC, then (by 80) $\measuredangle C \equiv \measuredangle H$. If not on BC, suppose C' between C and A. Then (by 44) $\measuredangle BC'A \equiv \measuredangle H$, and $BC' \equiv GH \equiv BC$. $\therefore$ (by 57) $\measuredangle BC'C \equiv \measuredangle C$.

176. Corollary to 175. Two triangles are congruent if they have two sides and the angle opposite the greater respectively congruent.

177. Definition. A triangle one of whose angles is a right angle is called a *right-angled* triangle, or more briefly a *right* triangle. The side opposite the right angle is called the *hypothenuse.*

178. Corollary to 176. Two right-angled triangles are congruent if the hypothenuse and one side are respectively congruent.

Ex. 161. If two triangles have two sides of the one respectively congruent to two sides of the other, and the angles opposite one pair of congruent sides congruent, then if these angles be not acute the triangles are congruent.

Ex. 162. If two triangles have two sides of the one respectively congruent to two sides of the other, and the angles opposite one pair of congruent sides congruent, then if one of the angles opposite the other pair of congruent sides is a right angle the triangles are congruent.

Ex. 163. If two triangles have two sides of the one respectively congruent to two sides of the other, and the angles opposite one pair of congruent sides congruent, then if the side opposite the given angle is congruent to or greater than the other given side the triangles are congruent.

Ex. 164. If any triangle has one of the following properties it has all:

1. Symmetry.
2. Two congruent sides.
3. Two congruent angles.
4. A median which is an altitude.
5. A median which is an angle-bisector.
6. An altitude which is an angle-bisector.
7. A perpendicular side-bisector which contains a vertex.
8. Two congruent angle-bisectors.

Ex. 165. The difference of any two sides of a triangle is less than the third side.

Ex. 166. From the ends of a side of a triangle the two sects to a point within the triangle are together less than the other two sides of the triangle, but make a greater angle.

Ex. 167. Two obliques from a point making congruent sects from the perpendicular are congruent, and make congruent angles with the straight.

Ex. 168. Of any two obliques between a given point and straight that which makes the greater sect from the foot of the perpendicular is the greater.

Ex. 169. Of sects joining two symmetrical points to a third, that cutting the axis is the greater.

179. Theorem. *If two triangles have two sides of the one respectively congruent to two sides of the other, then that third side is the greater which is opposite the greater angle.*

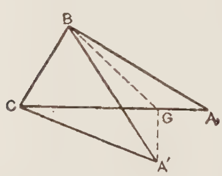

FIG. 76.

Proof. Take the triangles with one pair of congruent sides in common, BC, and on the same side of BC the other pair of congruent sides, BA, BA'. The bisector of $\measuredangle ABA'$, being within $\measuredangle ABC$, meets AC at a point G. Then (by 43) $\triangle ABG \cong \triangle A'BG$. $\therefore AG$

$\equiv A'G$. But (by 174) $A'G$ and GC are together greater than $A'C$.

179^b. Inverse of 179. If two triangles have two sides of the one respectively congruent to two sides of the other, then, of the angles opposite their third sides, that is the greater which is opposite the greater third side.

Ex. 170. Two right triangles are congruent if the hypothenuse and an acute angle are congruent, or if a perpendicular and an acute angle are congruent to a perpendicular and the corresponding acute angle.

Ex. 171. Given AB a sect, C its bisection-point, $PA \equiv PB$.

Prove $PC \perp AB$.

Ex. 172. Inverse. Given $CP \perp$ bi' of AB. Prove $PA \equiv PB$.

Ex. 173. Given $PM \perp AM \equiv PN \perp AN$. Prove $\measuredangle PAM \equiv \measuredangle PAN$ or its complement.

Ex. 174. Inverse. Given $\measuredangle PAM \equiv \measuredangle PAN$. Prove $PM \perp AM \equiv PN \perp AN$.

180. Definition. If AB is a diameter of a circle with center C, then the two points of the circle on any other diameter, being on opposite sides of C, are (by 25) on opposite sides of the straight AB. Hence the points of the circle other than the points A, B, are separated by AB into two classes of points uniquely paired. One of these classes together with the point A is called a *semicircle*. The other, with B, is the associated semicircle; A and B are called end-points of each semicircle.

181. Definition. If A, D are two points on the circle with center C, then, since (by 142) the end of

the perpendicular from C to the straight AD falls within a radius, therefore the points of the circle are not all on the same side of the secant AD. Hence the points of the circle other than the points A, D are separated by AD into two classes of points. One of these classes, together with the point A, is called an *arc*. The other, with the point D, is called the associated or *explemental* arc. A and D are called end-points of each arc.

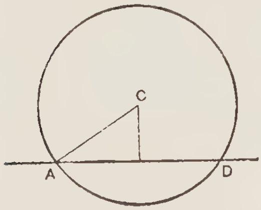

Fig. 77.

Of these two arcs the arc on the side of AD remote from the center is called the *minor* arc. The arc on the same side of AD as the center is called the *major* arc. The chord AD is said to be the chord of each of the two arcs. Thus to every arc pertains a chord, and to every chord pertain two arcs.

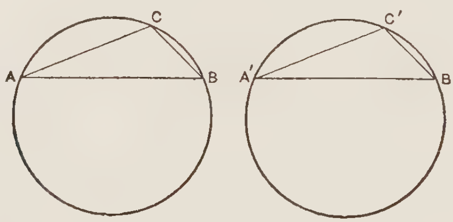

Fig. 78.

182. Definition. Two arcs AB, $A'B'$, are called congruent when, the end-points being mated, to every point C of the first arc corresponds one, and only one, point C' of the second, such that $AC \equiv A'C$ and $BC \equiv B'C'$.

183. Corollary to 182. Congruent arcs have congruent chords.

184. Definition. An angle having its vertex at the center of the circle is called an *angle at the center*, and is said to *intercept* the arc and chord, whose end points are on the angle's sides and whose other points are within the angle.

185. Theorem. *In a circle or in circles with congruent radii, congruent angles at the center intercept congruent arcs.*

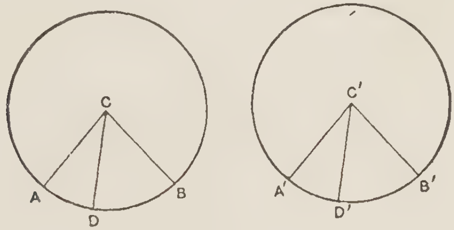

FIG. 79.

Given $\measuredangle ACB \equiv \measuredangle A'C'B'$.

To prove the minor arc $AB \equiv$ minor arc $A'B'$.

Proof. Since (by 43) $\triangle ACB \equiv \triangle A'C'B'$; $\therefore AB \equiv A'B'$.

Moreover, if D is any point within arc AB then ray CD is within $\measuredangle ACB$. Hence (by 48) there is within $\measuredangle A'C'B'$ a ray $C'D'$ meeting arc $A'B'$ in D', which makes $\measuredangle A'C'D' \equiv \measuredangle ACD$ and $\measuredangle B'C'D' \equiv \measuredangle BCD$. $\therefore$ (by 43) $A'D' \equiv AD$ and $B'D' \equiv BD$.

Also any point D'' of the minor arc $A'B'$ such that $A'D'' \equiv AD$ would (by 58) be on the ray making $\measuredangle A'C'D'' \equiv \measuredangle ACD \equiv \measuredangle A'C'D'$, and hence identical with D'.

186. Corollary. Any arc may be bisected.

187. Theorem. *Any two congruent arcs have congruent radii.*

Given arc $AMB \equiv$ arc $A'M'B'$.

To prove $CA \equiv C'A'$.

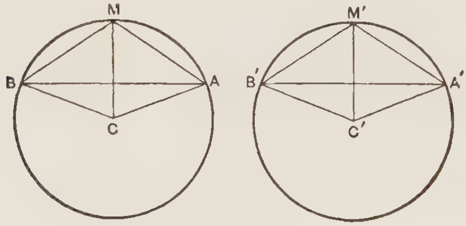

FIG. 80.

Proof. The bisector of $\measuredangle ACB$ cuts arc AMB in a point M such that, (by 43) $\triangle ACM \equiv \triangle BCM$. $\therefore AM \equiv BM$ and $\measuredangle BMC \equiv \measuredangle AMC$. From hypothesis there is a point M' of arc $A'M'B'$ such that $\triangle A'M'B' \equiv \triangle AMB$. $\therefore A'M' \equiv B'M'$.

$\therefore$ (by 58) $\triangle A'M'C' \equiv \triangle B'M'C'$.

$\therefore \measuredangle A'M'C' \equiv \measuredangle B'M'C'$.

$\therefore$ (by 48 and 84) $\measuredangle A'M'C' \equiv \measuredangle AMC$.

$\therefore$ (by 44) the two isosceles triangles $\triangle AMC \equiv \triangle A'M'C'$. $\therefore AC \equiv A'C'$.

188. Inverse of 185. Congruent minor arcs are intercepted by congruent angles at the center.

Proof. Since from hypothesis chord $AB \equiv$ chord $A'B'$, $\therefore$ (by 187 and 58) $\triangle ACB \equiv \triangle A'C'B'$.

$\therefore \measuredangle ACB \equiv \measuredangle A'C'B'$.

189. Theorem. *In a circle or in circles with congruent radii, congruent chords have congruent minor arcs.*

For the angles at the center on the congruent chords are congruent (by 58) [$\triangle$s with 3 sides $\equiv$]. $\therefore$ (by 185) the minor arcs they intercept are congruent.

190. Theorem. *Given a minor arc and a circle of congruent radius. There are on the circle two and only two arcs with a given end-point, congruent to the given arc.*

Proof. An angle at the center which intercepts the given arc can be set off (by III 4) once and only once on each side of the radius to the given point.

191. Theorem. *From any point of a circle there are not more than two congruent chords, and the chords are congruent in pairs, one on each side of the diameter from that point.*

Proof. If AB is any chord, take at center C on the other side of AC, the $\measuredangle ACB' \equiv \measuredangle ACB$, $\therefore$ by 43, $\triangle ACB' \equiv \triangle ACB$.

$\therefore AB' \equiv AB$.

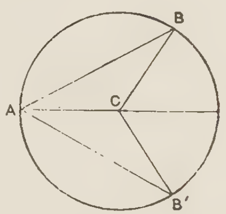

FIG. 81.

Moreover, were B'' the end-point of a third chord from A congruent to AB and to AB', then B, B', B'' would be at once on $\odot C(CA)$ and $\odot A(AB)$, which, by 122, is impossible.

192. Definition. If all the points of one arc are points of a second, but the second has also points not on the first, then the second is said to be *greater* than the first and any arc congruent to the second is said to be greater than any arc congruent to the first.

193. Theorem. *In a circle or in circles with congruent radii, of two angles at the center, the greater intercepts the greater arc and chord.*

Hypothesis. $CA \equiv C'A'$. $\measuredangle ACD > \measuredangle A'C'B'$.

Conclusion. Arc $AD >$ arc $A'B'$.

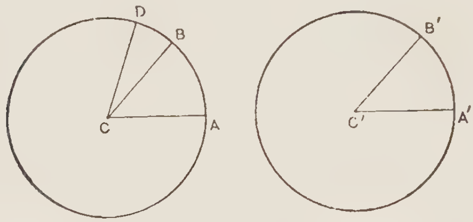

FIG. 82.

Proof. From C against CA toward D, (by III 4) take $\measuredangle ACB \equiv \measuredangle A'C'B'$. Then from hypothesis ray CB is within $\measuredangle ACD$.

$\therefore B$ is within arc AD.

$\therefore$ (by 192) arc $AD >$ arc AB. But (by 185) arc $A'B' \equiv$ arc AB. $\therefore$ (by 192) arc $AD >$ arc $A'B'$.

Moreover $\triangle A'C'B'$ has two sides $C'A'$, $C'B' \equiv CA$, CD of $\triangle ACD$, but $\measuredangle ACD > \measuredangle A'C'B'$, $\therefore$ (by 179) $AD > A'B'$.

194. Inverse of 193. In a circle or in circles with congruent radii the greater chord has the greater angle at the center and the greater minor arc.

For (by 179^b) it has the greater angle at the center, and $\therefore$, by 193, the greater minor arc.

195. Inverse of 193. In a circle or in circles with congruent radii, the greater minor arc has

the greater angle at the center and the greater chord.

196. Theorem. In a circle or in circles with congruent radii, *congruent chords have congruent perpendiculars from the center, and the lesser chord has the greater perpendicular.*

Proof. Of two chords from A on the same side of the diameter AC, one, say AD, is (by III 4) without the angle CAB made by the other, and hence its end-point D is on the minor arc AB. Hence (by 195) $\measuredangle ACB > \measuredangle ACD$ and $AB > AD$.

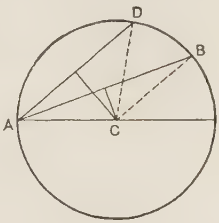

FIG. 83.

Moreover, the sect from the center to the bisection-point of AD, since D and so every point of AD is on the opposite side of AB from C, crosses the straight AB and $\therefore$ (by 142) is $>$ the perpendicular from C to AB.

Moreover, congruent chords anywhere have congruent perpendiculars (by 178).

197. Inverse of 196. In a circle or in circles with congruent radii, chords having congruent perpendiculars from the center are congruent, and the chord with the greater perpendicular is the lesser. For (by 196) it cannot be greater nor congruent.

Two Circles.

198. A figure formed by two circles is symmetrical with regard to their center-straight as axis.

Every chord perpendicular to this axis is bisected by it.

If the circles have a common point on this straight they cannot have any other point in common, for any point in each has in that its symmetrical point with regard to this axis, and circles with three points in common are identical.

199. Two circles with one and only one point in common are called tangent, are said to touch, and the common point is called the point of tangency or contact.

200. If two circles touch, then, since there is only one common point, this point of contact is on the center-straight, and a perpendicular to the center-straight through the point of contact is a common tangent to the two circles.

Ex. 175. Two circles cannot mutually bisect.

Ex. 176. The chord of half a minor arc is greater than half the chord of the arc.

Ex. 177. In a circle, two chords which are not both diameters do not mutually bisect each other.

Ex. 178. All points in a chord are within the circle.

Ex. 179. Through a given point within a circle draw the smallest chord.

Ex. 180. Rays from center to intersection points of a tangent with $\parallel$ tangents are $\perp$.

Ex. 181. A circle on one side of a triangle as diameter passes through the feet of two of its altitudes.

Ex. 182. In $\dagger \triangle ABC$ if D on $AB \equiv BC$, prove $CD > AD$.

Ex. 183. A circumscribed parallelogram is a rhombus.

Ex. 184. In $\triangle ABC$, having $AB > BC$, the median BD makes $\measuredangle BDA$ obtuse.

Ex. 185. If AB, a side of a regular $\triangle$, be produced to D, then $AD > CD > BC$.

Ex. 186. If BD is bisector t_b, and $AB > BC$, then $BC > CD$.

Ex. 187. How must a straight through one of the common points of two intersecting circles be drawn in order that the two circles may intercept congruent chords on it?

Ex. 188. Through one of the points of intersection of two circles draw the straight on which the two circles intercept the greatest sect.

Ex. 189. If any two straights be drawn through the point of contact of two circles, the chords joining their second intersections with each circle will be on parallels.

Ex. 190. To describe a ⊙ which shall pass through a given point, and touch a given ⊙ in another given point.

Ex. 191. To describe a ⊙ which shall touch a given ⊙, and touch a given st' [or another given ⊙] at a given point.

Ex. 192. The foot of an altitude bisects a sect from orthocenter to circum-⊙.

Ex. 193. If from the end-points of any diameter of a given ⊙ ⊥s be drawn to any secant their feet give with the center ≡ sects.

Ex. 194. A, B, I, I_c are concylic.

Ex. 195. If t_b meets circum-⊙ in D, then $DA \equiv DC \equiv DI$.

Ex. 196. The ⊥s at the extremities of any chord make ≡ sects on any diameter.

Ex. 197. If in any 2 given tangent ⊙s be taken any 2 ∥ diameters, an extremity of each diameter, and the point of contact shall be costraight.

Ex. 198. If 2 ⊙s touch internally, on any chord of one tangent to the other the point of contact makes sects which subtend ≡ ∡s at the point of tangency of the ⊙s.

Ex. 199. $2m_a > = < a$ according as ∡A acute, r't, obtuse.

Ex. 200. Chords joining the end-points of ∥ chords are ≡.

Ex. 201. St' through point of tangency meets ⊙ O at A, ⊙O' at A'. Prove $AO \parallel A'O'$.

Ex. 202. Intersecting ⊙s are † with regard to their center-st' and if ≡ are † with regard to their common chord.

Ex. 203. Find the bi' of an ∡ without using its vertex.

Ex. 204. A quad' with 2 sides ∥ and the others ≡ is either a ∥g'm or a symtra.

CHAPTER VIII.

A SECT CALCULUS.

201. On the basis of assumptions I 1, 2, and II–IV, that is, in the plane and without the Archimedes assumption, we will establish a sect calculus or geometric algorithm for sects, where all the operations are identical with those for real numbers. The following proof is due to F. Schur.

202. (Pascal.) Let A, B, C and A', B', C' be two triplets of points situated respectively on two perpendiculars and distinct from their intersection point O'. If AB' is parallel to $A'B$ and BC' parallel to $B'C$, then is also AC' parallel to $A'C$.

Proof. Call D' the point where the perpendicular from B upon the straight $A'C$ meets the straight $B'A'$. Then C is the orthocenter of the triangle $BA'D'$; therefore $D'C \perp A'B$ and $\therefore \perp AB'$.

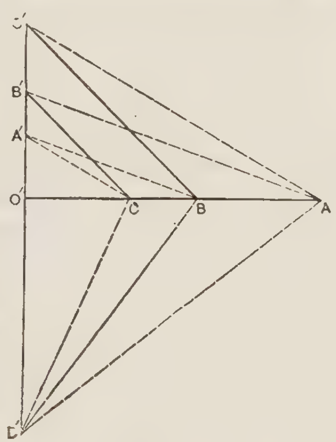

Fig. 84.

Consequently C is also the orthocenter of the tri-

angle $AB'D'$; $\therefore AD' \perp B'C$ and $\therefore$ also $\perp BC'$. Consequently B is the orthocenter of triangle $AC'D'$; $\therefore AC' \perp D'B$ and $\therefore AC' \parallel A'C$.

203. Instead of the word "congruent" and the sign $\equiv$, we use, in this sect calculus, the word "*equal*" and the sign $=$.

204. We begin by showing how from any two sects to find unequivocally a third by an operation we will call *addition*.

205. If A, B, C are three costraight points, and B lies between A and C, then we designate $c = AC$ as the *sum* of the two sects $a = AB$ and $b = BC$, and write to express this $c = a + b$.

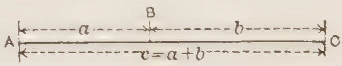

FIG. 85.

To add the two sects a and b in a determined order, we start from any point A, and take the point B such that $AB \equiv$, that is $= a$. Then on the straight AB beyond B we take the point C such that $BC = b$. Then the sect AC is what we have designated as the sum of the two sects $a = AB$ and $b = BC$ in the order $a + b$.

206. From III 3 follows immediately that this sum is independent of the choice of the point A, and independent of the choice of the straight AB.

By III 1, it is independent of the order in which the sects are added. Therefore $a + b = b + a$.

207. This is the *commutative* law for addition. Thus the commutative law for addition holds good,

is valid, for our summation of sects. But this law is not at all self-evident, and expresses no general magnitude relation, but a wholly definite geometric fact; for a, b are throughout not numbers, but only symbols for certain geometric entities, for sects.

208. The sects a and b are called *less* than C; in symbols: $a<c$, $b<c$; and c is called greater than a and b; in symbols $c>a$, $c>b$.

209. To add to $a+b$ a further sect c, take on straight AB beyond C the sect $CD=c$. Then the sect $AD=(a+b)+c$. But this same sect AD is, by the given definition of sum, also the sum of the sects AB and BD, that is of the sects a and $(b+c)$.

Thus $a+(b+c)=(a+b)+c$, and so is verified and valid what is called the *associative* law for addition.

210. To define geometrically the *product* of a sect a by a sect b, we employ the following construction. We choose first an arbitrary sect, which remains the same for this whole theory, and designate it by 1. This we set off from their intersection point on one of two perpendicular straights. On the other we set off on one ray a, on the other b. The circle through the free end-points of 1, a, b determines on the fourth ray a sect c. Then we name this sect c the product of the sect a by the sect b; and we write $c=ab$.

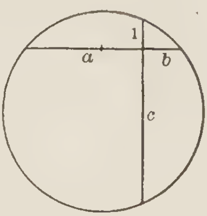

FIG. 86.

By $\equiv \triangle$s and 133, $ab=ba$. This is the *commutative* law for multiplication.

211. Considering the triangle of the end-points of

1 and a, it is equiangular to that of the end-points of b and c. This gives as an easy construction for our sect product the following:

Set off on one side of a right angle, starting from the vertex O, first the sect 1 and then, likewise from the vertex O, the sect b. Then set off on the other side the sect a. Join the end-points of the sects 1 and a, and draw a parallel to this straight through the end-point of the sect b. The sect which this parallel determines on the other side is the product ab; or we may call it ba, since, as we have already seen, $ab = ba$, which is also given by the fact that the triangle of the end-points of 1 and b is equiangular to that of the end-points of a and c.

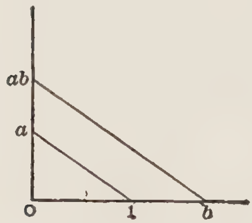

FIG. 87.

212. We emphasize that this definition is *purely geometric;* ab is not at all the product of two numbers.

213. To prove for our multiplication the *asso-*

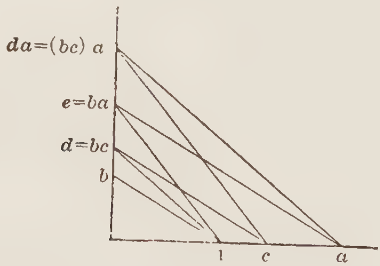

FIG. 88.

ciative law for multiplication $a(bc)=(ab)c$ we construct first the sect $d=bc$, then da, further the sect $e=ba$, and finally ec. The end-points of da and ec coincide (by Pascal), and by the commutative law follows the above formula for the associative law of sect multiplication.

214. Finally is valid in our sect-calculus also the *distributive* law $a(b+c)=ab+ac$.

To demonstrate it we construct the sects ab, ac, and $a(b+c)$, and draw through the end-point of the sect c (see Fig. 89) a parallel to the other side of the right angle. The congruence of the two right-angled triangles shaded in the figure and the application of the theorem of the equality of

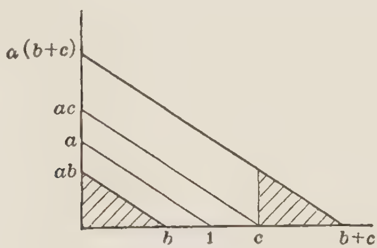

FIG. 89.

opposite sides in the parallelogram give then the desired proof.

215. If b and c are any two sects, there is always one and only one sect a such that $c=ab$; this sect a is designated by the notation $\frac{c}{b}$, and is called the *quotient* of c by b.

The Sum of Arcs.

216. Definition. If a and b are two arcs with equal radii, their sum, $a+b$, is the arc obtained by taking together as one arc the arc a and an arc congruent to b having as one of its end-points an end-point of a and its points taken as outside of a.

217. Theorem. In the same circle or in circles with equal radii, if minor arc $a \equiv$ minor arc a' and

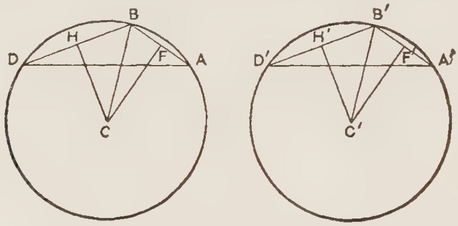

FIG. 90.

minor arc $b \equiv$ minor arc b', then arc $(a+b) \equiv$ arc $(a'+b')$.

Let minor arc $AB=a$ and minor arc $BD=b$, minor arc $A'B'=a'$ and minor arc $B'D'=b'$.

To prove arc $ABD \equiv$ arc $A'B'D'$.

Proof. $\triangle CBF \equiv \triangle C'B'F'$ (two sides and included $\measuredangle$) $\therefore \measuredangle CBF \equiv \measuredangle C'B'F'$. In same way $\measuredangle CBH \equiv \measuredangle C'B'H'$. $\therefore$ (by 49) $\measuredangle HBF \equiv \measuredangle H'B'F'$. $\therefore$ (two sides and included $\measuredangle$) chord $AD=$ chord $A'D'$. $\therefore$ (by 189) minor arc $AD \equiv$ minor arc $A'D'$; and if $a+b$ is minor, so (by inverse of 133) is $a'+b'$. But if $a+b$ be not a minor arc, then if P be any point on the semicircle or major arc $a+b$, take $\measuredangle D'C'P' \equiv \measuredangle DCP$ with P' on the same side of

$D'C'$ with reference to A' as P of DC with reference to A. Thus $D'P' = DP$, also $\measuredangle DPA \equiv \measuredangle D'P'A'$

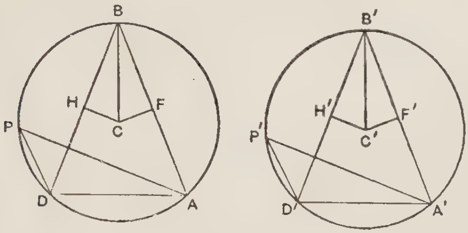

FIG. 91.

and $\measuredangle DAP \equiv \measuredangle D'A'P'$, since A and A' are on the same side of DP and $D'P'$. $\therefore \triangle APD \equiv \triangle A'P'D'$.

218. Definition. If an angle at the center is right the arc it intercepts is called a *quadrant*.

219. Corollary to 217. In a circle or in circles with equal radii the sum of any two quadrants is congruent to the sum of any other two, and all semicircles are congruent.

A circle is the sum of two semicircles or four quadrants.

Congruent major arcs are the sums of congruent semicircles and congruent minor arcs.

220. Convention. We may look upon a semicircle as an arc whose chord is a diameter, and we may look upon a whole circle as a major arc whose two end-points coincide. The explemental minor arc will then be one single point.

We may even think of arcs on a circle greater than the whole circle. In such a case certain points on the circle are considered more than once.

221. Any arc may now be expressed as a sum of a number of quadrants and a minor arc.

The Sum of Angles.

222. Definition. The sum of two acute angles or of a right angle and an acute angle is the angle obtained by setting off one against the other from its vertex with no interior point in common, and then omitting the common ray.

The sum of any two or more angles is an aggregate of right angles and one acute angle such that, taken as angles at the center of any one circle, the sum of the intercepted quadrants and the arc intercepted by the acute angle equals the sum of the arcs intercepted by the angles to be added together.

223. Corollary to 79. The sum of the three angles of any rectilineal triangle is two right angles.

The sum of two supplemental angles is two right angles.

224. In the familiar terminology of motion circles with equal radii are called congruent, and we say they can be made to coincide if the center of one be placed on the center of the other.

Since, in their congruence, any one given point of the one can be mated with any point of the other, we say, after coincidence the second circle may be turned about its center, and still coincide with the first.

Hence also a circle can be made to slide along itself by being turned about its center. This expresses a fundamental characteristic of the circle. It allows us to turn any figure connected with the circle about the center without changing its relation to the circle. Such displacement is called a *rotation*.

A displacement of a figure connected with a

straight, in which the straight slides on its trace, is called *translation.*

That translation can be effected without rotation is an assumption about equivalent to the parallel Assumption IV.

225. Theorem. The diameter perpendicular to a chord bisects the angle at the center, and the two arcs, minor and major, made by the chord.

226. Convention. Parallel secants or parallel chords are said to intercept the two arcs whose points are between the parallels.

227. Theorem. *Parallel chords intercept congruent arcs.*

Given $AB \parallel A'B'$.

To prove minor $AA' \equiv$ minor arc BB'.

Proof. If $CD \perp AB$ then also (by 74) $CD \perp A'B'$. Then (by 117 and 58) $\measuredangle ACD \equiv \measuredangle BCD$ and $\measuredangle A'CD \equiv \measuredangle B'CD$. $\therefore$ (by 49) $\measuredangle ACA' \equiv \measuredangle BCB'$. $\therefore$ (by 185) minor arc $AA' \equiv$ minor arc BB'.

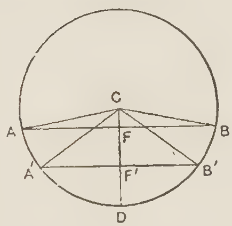

FIG. 92.

228. Theorem. *If a simple plane polygon be cut into triangles by diagonals within the polygon the sum of their angles, together with four right angles, equals twice as many right angles as the polygon has sides.*

FIG. 93.

Proof. By a diagonal within the polygon cut off a triangle. This diminishes the number of sides by one and the sum of the angles by two right angles. So reduce the sides

to three. We have left two more sides than pairs of right angles.

229. Definition. The exterior angle at any vertex of a polygon is the angle between a side and the ray made by producing the other side through the vertex.

230. Theorem. *In any convex plane polygon the sum of the exterior angles, one at each vertex, is four right angles.*

Proof. The exterior angle is the supplement of the adjacent angle in the polygon. This pair gives a pair of right angles for every side. But (by 228) the angles of the polygon give a pair of right angles for every side except two.

Ex. 205. In r't $\triangle$, h_c makes $\measuredangle$s $\equiv$ to α and β.

Ex. 206. Always $m_c < \frac{1}{2}(a+b)$.

Ex. 207. From point without acute $\measuredangle$ α, $\perp$s to sides make $\measuredangle \equiv \alpha$.

Ex. 208. The sect joining the bisection-points of the non-$\parallel$ sides of a trapezoid is $\parallel$ to the $\parallel$ sides and half their sum.

Ex. 209. How many sides has a polygon, the sum of whose interior $\measuredangle$s is double the sum of its exterior $\measuredangle$s?

Ex. 210. How many sides has a regular polygon, four of whose $\measuredangle$s are together 7 r't $\measuredangle$s?

Ex. 211. The trisection-points of the sides of an equilateral $\triangle$ form a regular hexagon.

Ex. 212. The $\perp$s from A and B upon m_c are $\equiv$.

Ex. 213. Find the sum of 3 non-consecutive $\measuredangle$s of an inscribed hexagon.

Ex. 214. Construct $\dotplus\triangle$ from b and $a+h_b$; from β (or h_b) and perimeter.

Ex. 215. The sum of the three sects from any point within a $\triangle$ to the vertices is $<$ the sum and $> \frac{1}{2}$ sum of the 3 sides.

Ex. 216. Construct ⫞ r't △ from $b+c$.

Ex. 217. In a given st' find a point to which sects from 2 given points have the least sum.

Ex. 218. The sum of the medians in △ is < the sum and > ½ sum of sides.

Ex. 219. Construct △ from $\alpha-\beta$ and c. △ from α, β, r; △ from α, β, R.

Ex. 220. The sum of 2 opposite sides of a circumscribed quad' is half the perimeter. The sum of the ∡s they subtend at the center is 2 r't ∡s.

Ex. 221. In r't△, $a+b=c+2r=2R+2r$.

Ex. 222. From the vertices of △ as centers find 3 radii which give ⊙s tangent, two and two.

Ex. 223. If H is orthocenter, the 4 circum-⊙s of A, B, C, H are ≡.

Ex. 224. Of I, I_1, I_2, I_3, each is the orthocenter of the other 3, and the 4 circum-⊙s are ≡.

Ex. 225. If, of a pentagon, the sides produced meet, the sum of the ∡s formed is 2 r't ∡s.

Ex. 226. If h_b meets b at D, construct △ from h_b, $a-AD$, $c-CD$.

Ex. 227. A quad' is a trapezoid if an opposite pair of the 4 △s made by the diagonals are ≡.

CHAPTER IX.

PROPORTION AND THE THEOREMS OF SIMILITUDE.

231. With help of the just-given sect-calculus Euclid's theory of proportion can in the following manner be established free from objection and without the Archimedes assumption.

232. Convention. If a, b, a', b' are any four sects, then the *proportion* $a : b = a' : b'$ shall mean nothing but the sect equation $ab' = ba'$.

233. Definition. Two triangles are called *similar* if their angles are respectively congruent. Sides between vertices of congruent angles are called *corresponding*.

234. Theorem. *In similar triangles the sides are proportional.*

Given a, b and a', b' corresponding sides in two similar triangles.

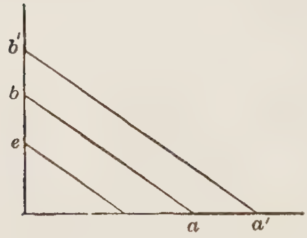

Fig. 94.

To prove the proportion $a : b = a' : b'$.

Proof. We consider first the special case, where the angles included by a, b and a', b' in the two triangles are right, and suppose both triangles on the same right angle. We

then set off from the vertex on one side the sect 1, and take through the end-point of sect 1 the parallel to the two hypothenuses. This parallel determines on the other side the sect e. Then is, by our definition of the sect-product, $b=ea$, $b'=ea'$. Consequently we have $ab'=ba'$, that is, $a:b=a':b'$.

We pass now to the general case. Construct in each of the two similar triangles the in-center I, respectively I', and drop from these the three perpendiculars r, respectively r', on the triangle's sides. Designate the respective sects so determined on the sides of the triangles by a_b, a_c, b_c, b_a, c_a, c_b, respectively a_b', a_c', b_c', b_a', c_a', c_b'. The just-proven special case of our theorem gives then the proportions

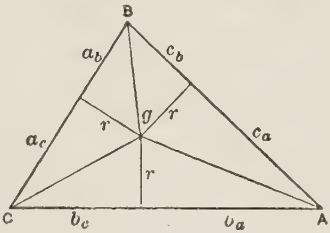

FIG. 95.

$$a_b : r = a_b' : r', \quad b_c : r = b_c' : r',$$
$$a_c : r = a_c' : r', \quad b_a : r = b_a' : r'.$$

From these we conclude by the distributive law

$$a : r = a' : r', \qquad b : r = b' : r',$$

and consequently, in virtue of the commutative law of multiplication,

$$a : a' = r : r' = b : b', \quad \text{and} \quad a : b = a' : b'.$$

235. From the just-proven theorem (234) we get easily the fundamental theorem of the theory of proportion, which is as follows:

Theorem. If two parallels cut off on the sides of any angle the sects a, b, respectively a', b', then holds good the proportion $a : b = a' : b'$. Inversely, when four sects a, b, a', b' fulfill this proportion, if the pairs a, a' and b, b' are set off upon the respective sides of any angle, then the straight joining the end-points of a and b is parallel to that joining the end-points of a' and b'.

Proof. First, since parallels make with the sides of the given angle similar triangles, therefore (by 234) $a : b = a' : b'$.

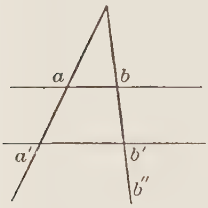

Fig. 96.

Second, for the inverse. Through the end-point of a' draw a parallel to the straight joining the end-points of a and b, and call the sect it determines on the other side b''. Then by First $a : b = a' : b''$. But by hypothesis $a : b = a' : b'$. $\therefore b'' = b'$.

236. Thus we have founded with complete rigor the theory of proportion on the basis of the Assumption-groups I–IV.

237. Corollary to 235. If straights are cut by any number of parallels the corresponding intercepts are proportional.

238. Corollary to 234. Parallels are divided proportionally by any three copunctal transversals.

239. Corollary to 235. Two triangles are similar if they have two sides proportional and the included angles congruent.

240. Definition. A point P, costraight with AB, but without the sect AB, is said to divide the sect AB *externally* into the sects PA, PB.

241. Corollary to 235. A sect can be divided nternally or externally in proportion to any two unequal given sects. The point of internal division is unique; likewise the point of external division.

242. Theorem. *The bisector of any angle of a*

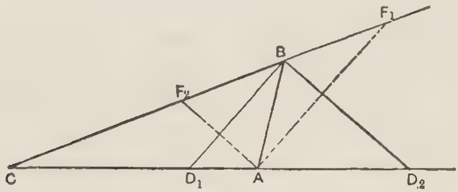

FIG. 97.

triangle or of its adjacent angle divides the opposite side in proportion to the other two sides.

[Proof. Take AF ‖ to bisector BD. Then $BF = c$.]

243. Definition. A sect divided internally and externally in proportion is said to be divided *harmonically*, and the four points are called a *harmonic range*.

244. Theorem. *A perpendicular from the right angle to the hypothenuse divides a right-angled triangle into two others similar to it, and is the mean proportional between the parts of the hypothenuse.*

Each side is the mean proportional between the hypothenuse and its adjoining part.

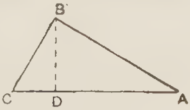

FIG. 98.

Proof. The r't $\triangle ABC \sim$ r't $\triangle ABD$, since $\measuredangle A$ is common.

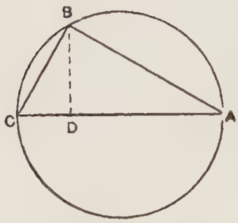

Fig. 99.

245. Corollary to 244. The perpendicular from any point in a circle to the diameter is the mean proportional between the parts of the diameter.

246. Theorem. *The square of the hypothenuse equals the sum of the squares of the two sides.*

Proof. $AC:AB = AB:AD$, that is, $AB^2 = AC \cdot AD$. Same way $BC^2 = AC \cdot DC$. Now add.

$$\therefore AB^2 + BC^2 = AC(AD + DC) = AC^2.$$

247. Theorem. *Triangles having their sides* taken in order *respectively proportional are similar.*

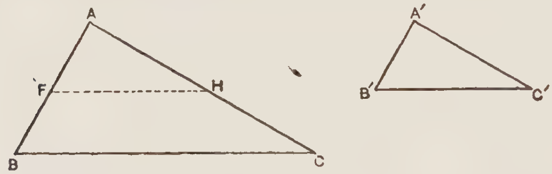

Fig. 100.

In the triangles ABC and $A'B'C'$ let $AB:A'B' = AC:A'C' = BC:B'C'$.

To prove that the triangles ABC and $A'B'C'$ are similar ($\sim$).

Proof. Upon AB take $AF = A'B'$, and upon AC take $AH = A'C'$. Then $AB:AF = AC:AH$. $\therefore$ (by 239) $\triangle ABC \sim$ to $\triangle AFH$. $\therefore AB:AF = BC:FH$. But by hypothesis $AB:AF = BC:B'C'$. $\therefore FH = B'C'$. $\therefore \triangle AFH \equiv \triangle A'B'C'$ [3 sides $\equiv$]. $\therefore \triangle ABC \sim \triangle A'B'C'$.

248. Theorem. *The product of the sects into which a given point divides chords of a given circle is constant.*

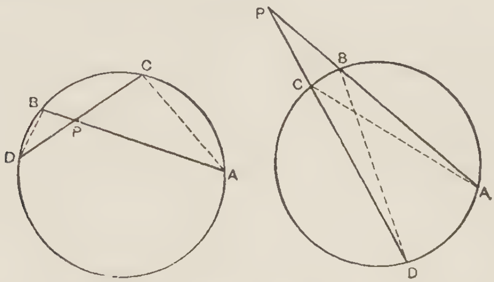

FIG. 101.

Hypothesis. Let chords AB and CD intersect in P.

Conclusion. $AP \cdot PB = CP \cdot PD$.

Proof. $\measuredangle PAC = \measuredangle PDB$ (by 133), and

$\measuredangle APC = \measuredangle BPD$; $\therefore \triangle APC \sim \triangle BPD$.

249. Corollary to 248. From a point taken on a tangent the square on the sect to the point of contact equals the product of the sects made on any secant.

The Golden Section.

250. Problem. *To divide a sect so that the product of the whole and one part equals the square of the other part.*

Required on AB to find P such that $AB \cdot PB = AP^2$.

Construction. Draw $BC \perp BA$ and $= \frac{1}{2}AB$ On the straight AC take D between A and C, and E beyond C such that $CD = CB = CE$. Take $AP = AD$

and $AP' = AE$. P and P' divide AB internally and externally in the golden section.

Proof. By 249, $AB^2 = AD \cdot AE = AP(AP + AB)$
$= AP^2 + AP \cdot AB.$

$\therefore AB(AB - AP) = AP^2$. $\therefore AB \cdot PB = AP^2$.

Again, $AB^2 = AE \cdot AD = P'A(AE - DE)$
$= P'A(P'A - AB) = P'A^2 - AB \cdot P'A.$
$\therefore AB(AB + P'A) = P'A^2.$
$\therefore AB \cdot P'B = P'A^2.$

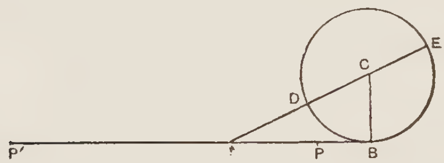

FIG. 102

251. Corollary to 250. If a is any sect divided in the golden section, its greater part $x = \frac{a}{2}\left[(5)^{\frac{1}{2}} - 1\right]$.

For (by 246) $AC^2 = AB^2 + BC^2 = a^2 + \frac{a^2}{4} = \left(\frac{a(5)^{\frac{1}{2}}}{2}\right)^2$

$$\therefore AP = AD = AC - CD = \tfrac{1}{2}a(5)^{\frac{1}{2}} - \frac{a}{2}.$$

252. Theorem. *The products of opposite sides of a non-cyclic quadrilateral are together greater than the product of its diagonals.*

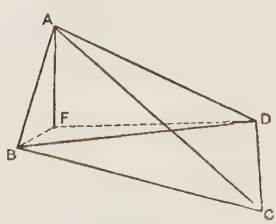

FIG. 103.

Proof. Make $\measuredangle BAF = \measuredangle CAD$, and $\measuredangle ABF = \measuredangle ACD$. Join FD.

Then $\triangle ABF \sim \triangle ACD$,
$\therefore BA : AC = FA \cdot AD.$

But this shows (since $\measuredangle BAC = \measuredangle FAD$), $\triangle BAC \sim \triangle FAD$.

From $\triangle ABF \sim \triangle ACD$, $\therefore AB \cdot CD = BF \cdot AC$.

From $\triangle BAC \sim \triangle FAD$, $\therefore BC \cdot AD = FD \cdot AC$.

$\therefore AB \cdot CD + BC \cdot AD = BF \cdot AC + FD \cdot AC > BD \cdot AC$.

253. Corollary to 252 (Ptolemy). The product of the diagonals of a cyclic quadrilateral equals the sum of the products of the opposite sides. (For then F falls on BD.)

254. Definition. Similar polygons are those of which the angles taken in order are respectively equal [*i.e.*, congruent], and the sides between the equal angles proportional.

255. Theorem. Two similar polygons can be divided into the same number of triangles respectively similar.

256. Theorem. If a cyclic polygon be equilateral it is regular.

Ex. 228. If AB is divided harmonically by P, P', then PP' is divided harmonically by A, B.

Ex. 229. If two triangles have the sides of one respectively parallel or perpendicular to the sides of the other they are similar.

Ex. 230. The corresponding altitudes of two similar triangles are proportional to any two corresponding sides.

Ex. 231. To divide a sect into parts proportional to given sects.

Ex. 232. A sect can be divided into any number of equal parts.

Ex. 233. To find the fourth proportional to three given sects.

Ex. 234. To find the third proportional to two given sects.

Ex. 235. If three non-parallel straights intercept proportional sects on two parallels they are copunctal.

Ex. 236. Every equiangular polygon circumscribed about a circle is regular.

Ex. 237. Every equilateral polygon circumscribed about a circle is regular if it has an *odd* number of sides.

Ex. 238. Every equiangular polygon inscribed in a circle is regular if it has an *odd* number of sides.

Ex. 239. One side of a △ is to either part cut off by a st' ∥ to the base as the other side is to the corresponding part.

Ex. 240. If a straight divides two sides of a △ proportionally, it is ∥ to the third side.

Ex. 241. The bisectors of an interior and an exterior ∡ at one vertex of a △ divide the opposite side harmonically.

Ex. 242. The perimeters of two ~ polygons are proportional to any two corresponding sides.

Ex. 243. A median and two sides of a trapezoid are copunctal.

Ex. 244. The chords on a st' through a contact-point of two ⊙s are proportional to their diameters; and a common tangent is a mean proportional between their diameters.

Ex. 245. The sum of the squares of the segments of 2 ⊥ chords equals the sq' of the diameter.

Ex. 246. On the piece of a tangent between two ∥ tangents the contact-point makes segments whose product is the square of the radius.

Ex. 247. To inscribe in and circumscribe about a given ⊙ a △~ to a given △.

Ex. 248. The hypothenuse is divided harmonically by any pair of st's through the vertex of the r't ∡ making ≡∡s with one of its sides.

Ex. 249. The bisection-point of the base of a △ and any point on a ∥ to the base through the vertex make a sect cut harmonically by a side and the other side produced.

Ex. 250. I divides t_b as b to $a+c$.

Ex. 251. $Rr:ac=b:2(a+b+c)$.

Ex. 252. In a r't △ the ⊥ sides are as the in-radii of △s made by h_c.

Ex. 253. Sects from the ends of the base of a △ to the intersections of a ∥ to base with the sides intersect on a median.

Ex. 254. △ from β, a/c, R.

Ex. 255. A quad' is cyclic if diagonals cut so that product of segments of one equals product of segments of the other.

Ex. 256. The sides of the pedal cut off △s ∼ to the original.

Ex. 257. Three points being given, to determine another, through which if any st' be drawn, ⊥s upon it from two of the former, shall together be equal to the ⊥ from the third.

Ex. 258. From two given sects to cut off two proportional to a second given pair so as to leave remainders proportional to another given pair.

Ex. 259. If one chord bisect another, and tangents from the extremities of each meet, the st' of their intersection points is ∥ to the bisected chord.

Ex. 260. In △, if sects from the ends of the base to the opposite sides intersect on the altitude, the joins of its foot to their ends will make equal angles with the base.

Ex. 261. The diagonals of a regular pentagon cut one another in the golden section, and the larger segments equal the sides.

Ex. 262. From the vertex of an inscribed △ a sect to the base ∥ to a tangent at either end of the base is a fourth proportional to the base and two sides.

Ex. 263. Straights from the vertices of any △ to the contact-points of the in-⊙ are copunctal.

Ex. 264. Construct △ from b, β, and that t_b makes segments as m to n.

Ex. 265. △ from β, m_b, and ∡ between b and m_b.

Ex. 266. △ from t_b and ⊥s on it from A and C.

Ex. 267. △ from β, $a-c$, and difference of segments made by h_b.

Ex. 268. R't △ from $a+b$, and $b+c$.

Ex. 269. △ from $\alpha-\beta$, $a:b=m:n$, and a third propor-

tional to *the difference* of segments made by h_c and *the lesser side.*

Ex. 270. △ from α, $a+b$, $a+c$.

Ex. 271. △ from α, R, and $b:c=m:n$.

Ex. 272. △ from α, b, $a-h_b$.

Ex. 273. Divide a given sect harmonically as m to n.

Ex. 274. In ∼ △s, $a:a'=h_a:h'_a'=m_a:m'_a'=t_a:t'_a'=r:r'=R:R'$.

Ex. 275. Two r't △s are ∼ if hypothenuse and a ⊥ are proportional.

Ex. 276. If a chord is bisected by another, either segment of the first is a mean proportional between the segments of the other.

Ex. 277. R't △ from a and the non-adjacent segment made by h_c.

Ex. 278. The diameter of a ⊙ is a mean proportional between the sides of the circumscribed regular △ and hexagon.

Ex. 279. From the center of a given ⊙ to draw a st' cutting off from a given tangent a sect any multiple of the segment between ⊙ and tangent.

Ex. 280. If 2 △s have two sides of the one proportional to two sides of the other, and ∡s, one in each, opposite one corresponding pair of these sides =, the ∡s opposite the other pair are either = or supplemental.

Ex. 281. The altitude to hypothenuse is a fourth proportional to it and the sides.

Ex. 282. The vertices of all △s on the same base with sides proportional are on a ⊙ with center costraight with base and radius a mean proportional between the sects from its center to the ends of base.

Ex. 283. To inscribe a sq. in a given △.

CHAPTER X.

EQUIVALENCE.

The theory of equivalence in the plane.

257. We take as basis for the investigations in the present chapter the Assumptions I, 1–2, and II–IV. We exclude the Archimedes assumption. Our theory of proportion and sect-calculus put us in position to found the Euclidean theory of equivalence by means of the assumptions named, that is, in the plane and independent of the Archimedes assumption.

258. Convention. If we join two points of a polygon P by any sect-train which runs wholly in the interior of the polygon we obtain two new polygons, P_1 and P_2, whose inner points all lie in the interior of P.

We say: P is separated or cut into P_1 and P_2; P_1 and P_2 *together compose* P.

259. Definition. Two polygons are called *equivalent* if they can be cut into a finite number of triangles congruent in pairs.

260. Definition. Two polygons are said to be *equivalent by completion* if it is possible so to annex

equivalent polygons to them that the two polygons so composed are equivalent.

261. We will use the sign of equality (=) between polygons to denote "equivalent by completion."

262. From these definitions follows immediately: By uniting equivalent polygons we get again *equivalent* polygons. If we take away equivalent polygons from equivalent polygons the remaining polygons are *equivalent by completion.*

Furthermore, we have the following propositions:

263. Theorem. Two polygons P_1 and P_2 equivalent to a third P_3 are equivalent.

Two polygons equivalent by completion to a third are equivalent by completion.

Proof. By hypothesis there is as well for P_1 as for P_2 an assignable partition into triangles such that each of these two partitions corresponds respectively to a partition of the polygon P_3 into congruent triangles.

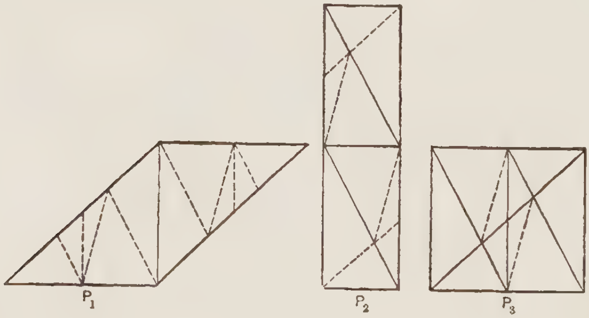

FIG. 104.

If we consider these two partitions of P_3 simultaneously every triangle of the one partition will in

general be cut into polygons by sects which pertain to the other partition. Now we introduce a sufficient number of sects to cut each of these polygons itself into triangles, and then make the two corresponding partitions into triangles in P_1 and in P_2. Then these two polygons P_1 and P_2 are cut into the same number of triangles congruent in pairs, and are therefore by our definition equivalent.

Again, if $Q_1 = Q_3$ and $Q_2 = Q_3$, then according to definition the composite $Q_1 + P_1$ is equivalent to $Q_3 + P_1$, and $Q_2 + P_2$ is equivalent to $Q_3 + P_2$. Therefore $Q_1 + P_1 + P_2$ is equivalent to $Q_3 + P_1 + P_2$, which is equivalent to $Q_2 + P_2 + P_1$. $\therefore Q_1 = Q_2$.

Parallelograms and Triangles with equal bases and altitudes.

264. Theorem. *Two parallelograms with equal bases and equal altitudes are equivalent by completion.*

Proof. $\triangle BAE \equiv \triangle CDF$. Annex $\triangle BCH$ and leave out $\triangle DHE$. $\therefore ABCD = EBCF$.

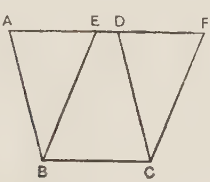

FIG. 105.

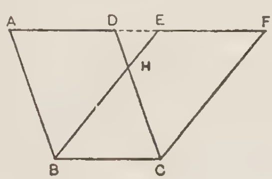

FIG. 106.

To prove these parallelograms equivalent would require here the Archimedes assumption.

265. Theorem. *Any triangle ABC is always equivalent to a certain parallelogram with equal base and half altitude.*

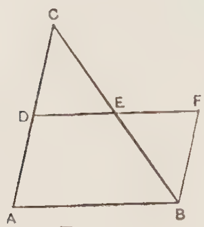

Fig. 107.

Proof. Bisect AC in D and BC in E, and then prolong DE to F, making $EF = DE$. The triangles DEC and FBE are then congruent, and consequently the $\triangle ABC$ and parallelogram $ABFD$ are equivalent.

266. From 264 and 265 follows with help of 263 immediately:

Theorem. *Two triangles of equal bases and equal altitudes are equivalent by completion.*

267. That two triangles with equal bases and altitudes are always equivalent cannot possibly be proven without using the Archimedes assumption.

268. The remaining theorems of elementary geometry about the equivalence by completion of polygons, and also, in particular, the Pythagoras equivalence theorem: "The square on the hypothenuse of a right triangle is equivalent to the united squares on the other two sides," are easy consequences of the theorems just set up.

269. But, nevertheless, in further working out the theory of equivalence we encounter an essential difficulty.

In particular our considerations hitherto leave undecided whether perhaps all polygons are not always equivalent by completion to one another. In this case all the previously established theo-

rems would teach nothing and be without importance.

The proven theorems about equivalence by completion are entirely rigorous; nevertheless we recognize on closer investigation that they all for the present have no content. We do not yet know whether there are polygons at all which are not equivalent by completion.

270. And not only must we know this, if we would undertake anything with our theorems, but also we need to consider the more specific question whether two rectangles equivalent by completion, having one common side, have also necessarily their other sides congruent, that is, whether a rectangle is uniquely determined by one of its sides and its equivalence by completion.

271. As the closer consideration shows, we need for answering the questions raised the inverse of 266, which runs as follows:

Theorem. *If two triangles equivalent by completion have equal bases then they have also equal altitudes.*

This fundamental theorem is the thirty-ninth of the first book of Euclid's Elements (Eu. I, 39). However, to prove it Euclid invokes the general theorem about magnitudes: "The whole is greater than its part," a procedure which amounts to the introduction of a new geometric assumption, that is, the tacit assumption of a new and independent magnitude, the "surface" or "superficial content."

272. Now for the question of superficial content, we can, on the basis of only our old assumptions, though into them the word "content" does not in

any way enter, prove that two polygons can be compared as to content.

273. Thus the congruence and equality of sects is fundamental or primitive, rooted immediately in assumptions.

274. But the equality of polygons as to content is a constructible idea with nothing new about it but a definition.

275. We proceed now to establish this theorem (Eu. I, 39) and therewith the theory of content in the way we desire, that is, merely with help of the plane assumptions without using the Archimedes assumption.

276. It need not surprise us that the proof is not wholly simple. For that two triangles are equivalent by completion according to definition only says that certain "corresponding" triangle-partitions exist; thereby can the number of the triangles be very great and one does not immediately see how from that we can conclude from equality of bases equality of the altitudes.

277. We begin by introducing the idea of area.

The area of triangles and polygons.

278. Definition. In any triangle ABC with the sides a, b, c, if we construct the two altitudes $h_a = AD$, $h_b = BE$, then follows from the similarity of the triangles BCE and ACD, (by 234) the proportion $a : h_b = b : h_a$, that is, $ah_a = bh_b$.

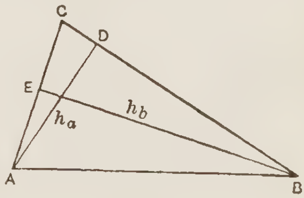

Fig. 108.

Consequently in every triangle the product of a base and its altitude is independent of what side of the triangle one chooses as base. Half the product of base and altitude of a triangle $\triangle$ is called the *area of the triangle* $\triangle$, and designated by $A(\triangle)$.

279. Convention. A sect which joins a vertex of a triangle with a point of the opposite side is called a *transversal;* this cuts the triangle into two triangles with common altitude, whose bases lie on the same straight. Such a partition is called a *transversal partition of the triangle.*

280. Theorem. *If a triangle $\triangle$ is in any way cut by any straights into a certain finite number of triangles $\triangle_k$, then is always the area of the triangle $\triangle$ equal to the sum of the areas of all the triangles $\triangle_k$.*

Proof. From the distributive law in our sect-calculus follows immediately that the area of any triangle is equal to the sum of the areas of two triangles which arise from the first by any transversal partition. Thus, for example,

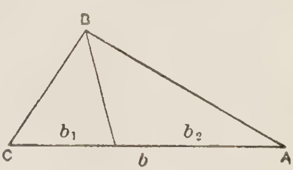

FIG. 109.

$$A(\triangle_1)+A(\triangle_2)=\tfrac{1}{2}b_1h+\tfrac{1}{2}b_2h=\tfrac{1}{2}h(b_1+b_2)=\tfrac{1}{2}bh=A(\triangle).$$

The repeated application of this fact shows that the area of any triangle is also equal to the sum of the areas of all the triangles which arise from the first, if we make successively however many transversal partitions.

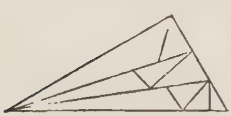

FIG. 110.

In order now to accomplish the corresponding demonstration for any partition of the triangle $\triangle$ into triangles $\triangle_k$, we draw from one vertex A of the triangle $\triangle$ through each dividing-point of the partition, that is, through each vertex of the triangles $\triangle_k$, a transversal; by these transversals the triangle $\triangle$ is cut into certain triangles $\triangle_t$. Each of these triangles $\triangle_t$ is cut by the dividing-sects of the given partition into certain triangles and quadrilaterals. If in each quadrilateral we draw a diagonal then each triangle $\triangle_t$ is cut into certain triangles $\triangle_{ts}$.

We will now show that the partition into triangles $\triangle_{ts}$, as well for the triangles $\triangle_t$ as also for the triangles $\triangle_k$, is a chain of transversal partitions.

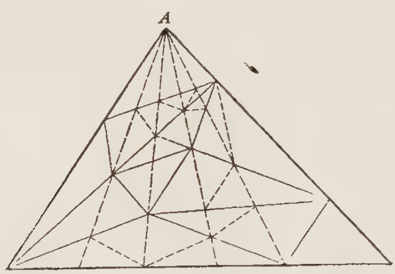

FIG. 111.

In fact, first is clear, that every partition of a triangle into part-triangles can always be effected by a series of transversal partitions, if, in the partition, no dividing-points lie in the interior of the triangle, and besides at least one side of the triangle remains free from dividing-points.

Now these conditions are evident for the triangles $\triangle_t$ from the circumstance that for each of them

the interior and one side, that opposite the point A, are free from dividing-points.

But also for every $\triangle_k$ is the partition into $\triangle_{ts}$ reducible to transversal partitions. In fact, if we consider a triangle $\triangle_k$, then there is, among the transversals from A in the triangle $\triangle$ a certain transversal which either cuts the triangle $\triangle_k$ into two triangles, or else upon which a side of $\triangle_k$ falls. For we recall that within no $\triangle_k$ are there dividing-points.

By construction, through every vertex of $\triangle_k$ goes a transversal from A; and there is always one vertex of $\triangle_k$ for which this transversal has a second point not in the region exterior to $\triangle_k$; it therefore either goes through the interior of $\triangle_k$ or upon it is a side of $\triangle_k$.

In this latter case this side of the triangle $\triangle_k$ remains altogether free from further dividing-points in the partition into triangles $\triangle_{ts}$. In the other case the sect of that transversal within the triangle $\triangle_k$ is for both the triangles thus arising a side which in the partition into triangles $\triangle_{ts}$ remains surely free from further dividing-points.

From the considerations at the beginning of this demonstration the area $A(\triangle)$ of the triangle $\triangle$ equals the sum of all areas $A(\triangle_t)$ of the triangles $\triangle_t$, and this sum is equal to the sum of all areas $A(\triangle_{ts})$. On the other hand is also the sum of the areas $A(\triangle_k)$ of all triangles $\triangle_k$ equal to the sum of all areas $A(\triangle_{ts})$. Hence, finally, the area $A(\triangle)$ is also equal to the sum of all areas $A(\triangle_k)$. So the theorem is completely proven.

281. Definition. If we define the area $A(P)$ of a polygon as the sum of the areas of all triangles into which it is cut in a certain partition, then the area of a polygon is independent of the way it is cut into triangles, and consequently determined uniquely simply by the polygon itself.

Proof. Suppose $\triangle_c$ to be the triangles of a certain partition, and $\triangle_k$ those of any other partition. Considering these two partitions simultaneously, in general is every triangle $\triangle_c$ cut into polygons by sects pertaining to $\triangle_k$. Now we introduce sects sufficient to cut these polygons themselves into triangles $\triangle_s$. Then the triangles $\triangle_c$ have (by 280) for the sum of their areas the sum of the areas of $\triangle_s$. But so also have the triangles $\triangle_k$.

[This fact, that the sum named is independent of the way of cutting up the polygon, is the kernel, the essence of this whole investigation.]

282. Corollary to 281. *Equivalent polygons have equal area.*

283. Moreover, if P and Q are two polygons equivalent by completion, then there must be, from the definition, two equivalent polygons P' and Q', such that the polygon compounded of P and P' is equivalent to the polygon compounded of Q and Q'. From the two equations

$$A(P+P')=A(Q+Q'), \quad A(P')=A(Q'),$$

we deduce at once $A(P)=A(Q)$, that is, *polygons equivalent by completion have equal area.*

284. From this latter fact we get immediately the proof of the theorem of 271 (Eu. I, 39). For,

designating the equal bases of the two triangles by b, the corresponding altitudes by h and h', we then conclude from the assumed equivalence by completion of the two triangles that they must also have equal area; that is, it follows $\frac{1}{2}bh=\frac{1}{2}bh'$, and, consequently, after division by $\frac{1}{2}b$, $h=h'$; which was to be proved.

Area and Equivalence-by-completion.

285. In what precedes we have found that polygons equivalent-by-completion have always equal area. The inverse is also true.

286. To prove the inverse, we consider first two triangles ABC and $AB'C'$ with a common right angle at A. The areas of these two triangles are expresesd by the formulas

$$A(ABC)=\tfrac{1}{2}AB\cdot AC,$$
$$A(AB'C'=\tfrac{1}{2}AB'\cdot AC'.$$

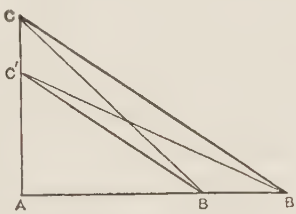

FIG. 112.

If we assume that these two areas are equal, we have

$$AB\cdot AC=AB'\cdot AC',$$

or $AB:AB'=AC':AC,$

and from this it follows (by 235) that the two straights BC' and $B'C$ are parallel, and then we recognize (from 266) that the two triangles $BC'B'$ and $BC'C$ are equivalent-by-completion. By annexing the triangle ABC' it follows that the two triangles ABC and $AB'C'$ are equivalent-by-completion. Thus we have proved that two right-

angled triangles with equal area are also always equivalent-by-completion.

287. Take now any triangle, with base b and altitude h, then this is (by 266) equivalent-by-completion to a right-angled triangle with the two perpendicular sides b and h; and since the original triangle evidently has the same area as the right-angled triangle, so it follows that in the preceding article the limitation to right-angled triangles was not necessary. Thus we have shown that *any two triangles with equal area are also always equivalent-by-completion.*

288. Now let P be any polygon with area b. Let P be cut into n triangles with the respective areas $b_1, b_2, \ldots b_n$; then is $b=b_1+b_2+\ldots+b_n$.

Construct now a triangle ABC with the base $AB=b$ and the altitude $h=1$ and mark on the base the points $A_1, A_2, \ldots A_n$, such that $b_1=AA_1$, $b_2=A_1A_2, \ldots b_{n-1}=A_{n-2}A_{n-1}$, $b_n=A_{n-1}B$. Since

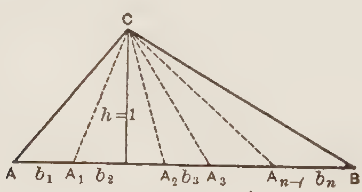

FIG. 113.

the triangles within the polygon P have respectively the same area as the triangles AA_1C, $A_1A_2C, \ldots,$ $A_{n-2}A_{n-1}C$, $A_{n-1}BC$, so they are, by what has just been proven, equivalent-by-completion to these.

Consequently the polygon P is equivalent-by-completion to a triangle with the base b and the altitude $h=1$.

Hence follows, with help of theorem 287, that two polygons of equal area are always equivalent-by-completion.

289. We may combine the two results found in this article 288 and in 283 into the following theorem: *Two polygons equivalent-by-completion have always the same area; and inversely, two polygons with equal area are always equivalent-by-completion.*

290. In particular two rectangles equivalent-by-completion which have one side in common must also have their other sides congruent.

291. Also follows the theorem: *If we cut a rectangle by straights into several triangles and leave out even one of these triangles, then we cannot with the remaining triangles fill out the rectangle.*

In what precedes is shown that this theorem is completely independent of the Archimedes assumption. Moreover, without the application of the Archimedes assumption, this theorem 291 does not suffice of itself for demonstrating Eu. I, 39.

292. Definition. Of two polygons P and Q, we call P *of lesser content* (respectively, *of equal*, *of greater content*) than Q, according as the area $A(P)$ is less (equal, greater) than $A(Q)$.

293. From what precedes it is clear that the concepts of equal content, of lesser content, of greater content are mutually exclusive.

294. Further, we see that a polygon which lies

wholly within another polygon must always be of lesser content than this latter.

295. Herewith we have established the essential theorems of the theory of superficial content, wholly upon considerations of the congruence of sects and angles, and without *assuming* superficial content to be a magnitude.

296. Theorem. *The area of any parallelogram is the product of the base by the altitude.*

297. Corollary to 296. The area of any rectangle or square is the product of two consecutive sides.

298. A square whose side is the unit sect has for area this unit sect,

since $$1 \times 1 = 1.$$

Any polygon has for area as many such unit sects as the polygon contains such squares on the unit sect.

The number expressing the area of a polygon will thus be the same in terms of our unit sect or in terms of a square on this sect considered as a new kind of unit, a unit surface, or unit of content. Such units, though traditional, are unnecessary and sometimes exceedingly awkward, as, for example, the acre.

Ex. 284. If twice the number expressing the area of a triangle be divided by the number expressing the base, the quotient is the number expressing the altitude.

Ex. 285. One side of a triangle is $35 \cdot 74$, and the altitude on it is $6 \cdot 3$. Find the area.

299. Theorem. *If two triangles* (or parallelograms)

have one angle of the one congruent to one angle of the other, their areas are proportional to the products of their sides about the congruent angles.

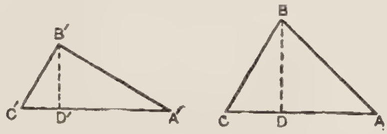

FIG. 114.

Let $\measuredangle C \equiv \measuredangle C'$,

then $\dfrac{\text{Area} \triangle ABC}{\text{Area} \triangle A'B'C'} = \dfrac{\frac{1}{2}AC \cdot BD}{\frac{1}{2}A'C' \cdot B'D'} = \dfrac{AC \cdot BD}{A'C' \cdot B'D'}$.

But in $\sim \triangle$'s BCD and $B'C'D'$, $\dfrac{BD}{B'D'} = \dfrac{BC}{B'C'}$.

$$\therefore \frac{\text{Area} \triangle ABC}{\text{Area} \triangle A'B'C'} = \frac{AC \cdot BC}{A'C' \cdot B'C'}.$$

300. Corollary to 299. The areas of similar triangles are proportional to the squares of corresponding sides.

301. Problem. To construct a rectangle, given two consecutive sides.

Construction. Take a straight and a perpendicular to it. From the vertex of the right angle take one given sect on the straight, the other on the perpendicular. Through their second end-points draw perpendiculars. These (by 77) meet. They intersect in the fourth vertex of the rectangle required.

FIG. 115.

Proof. By construction the figure is a parallelogram with one right angle; $\therefore$ a rectangle.

302. Corollary to 301. So we may construct a square on any given sect.

303. Theorem. *The square on the hypothenuse of any right-angled triangle is equivalent to the sum of the squares on the other two sides.*

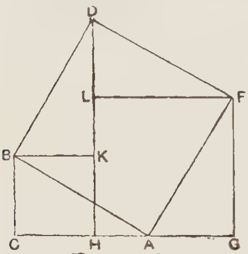

FIG. 116.

Hypothesis. $\triangle ABC$, r't-angled at C.

Conclusion. Square on AB is equivalent to sq. on AC + sq. on BC.

Proof. On hypothenuse AB, on side opposite C, construct (by 302) the sq. $ABDF$. From its vertices D, F drop DH, $FG \perp CA$, and $\therefore \parallel BC$. Drop BK, $FL \perp DH$, and $\therefore \parallel AC$. Then $\measuredangle ABC \equiv \measuredangle DBK$ (complements of $\measuredangle ABK$). Also $\measuredangle BDK \equiv \measuredangle DFL$ (complements of $\measuredangle LDF$). Also $\measuredangle DFL \equiv \measuredangle AFG$ (complements of $\measuredangle AFL$). $\therefore$ (by 44), $\triangle ABC \equiv \triangle DBK \equiv \triangle FDL \equiv \triangle FAG$. $\therefore$ $BCHK$ is sq. on BC, and $FGHL =$ sq. on AC. $\therefore$ sq. on $AB \equiv AFLKB +$ $2\triangle ABC \equiv$ sq. on $BC +$ sq. on AC.

304. Problem. *To construct an equilateral triangle on a given sect.*

Construction. On the st. AB from A take the given sect AB. At B erect to AB a perpendicular. On this perpendicular, from B take BC, the given sect. Join AC. At C erect to the straight AC the perpendicular CD. On CD from C take CD, the given sect. Join AD. Bisect AD at E, and AB at F. Erect at F to AB

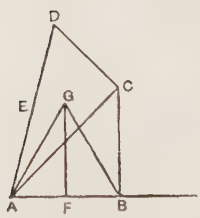

FIG. 117.

the perpendicular FG. From F take $FG=AE$. ABG is the required equilateral triangle.

Proof. $\overline{AG}^2=\overline{AF}^2+\overline{FG}^2=(\frac{1}{2}AB)^2+(\frac{1}{4}3(AB)^2=\overline{AB}^2$.

305. Theorem. *In any triangle, the square of a side opposite any acute angle is less than the sum of the squares of the other two sides by twice the product of either of those sides and a sect from the foot of that side's altitude to the vertex of the acute angle.*

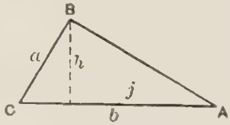

FIG. 118.

Proof. Let a, b, c denote the sides, and h denote b's altitude, and j the sect from its foot to the acute angle A.

$$a^2-h^2=(b-j)^2=b^2-2bj+j^2=b^2-2bj+c^2-h^2;$$
$$\therefore a^2=b^2-2bj+c^2.$$

306. Theorem. In an obtuse-angled triangle the square of the side opposite the obtuse angle is greater than the sum of the squares of the other two sides by twice the product of either of those sides and a sect from the foot of that side's altitude to the vertex of the obtuse angle.

Ex. 286. Find the area of an isosceles triangle whose base is 60 and each of the equal sides 50.

Ex. 287. If two triangles (or parallelograms) have an angle of one supplemental to an angle of the other, their areas are as the products of the sides including the supplementary angles.

Ex. 288. The area of any circumscribed polygon is half the product of its perimeter by the radius of the inscribed circle.

Ex. 289. To find the area of a trapezoid. Rule: Multiply the sum of the parallel sides (its bases) by half their common perpendicular (its altitude).

Ex. 290. The area of a trapezoid equals the product of its altitude by its median (the sect joining the bisection-points of the non-parallel sides).

307. *To find the altitudes of a triangle in terms of its sides.*

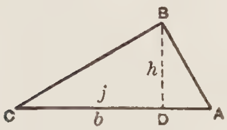

FIG. 119.

Either $\measuredangle A$ or $\measuredangle C$ is acute. Suppose $\measuredangle C$ acute,

$c^2 = a^2 + b^2 - 2bj$ (by 305).

$$\therefore j = \frac{a^2 + b^2 - c^2}{2b}.$$

$$h_b{}^2 = a^2 - j^2 = a^2 - \frac{(a^2+b^2-c^2)^2}{4b^2} = \frac{4a^2b^2 - (a^2+b^2-c^2)^2}{4b^2} =$$

$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4b^2} =$$

$$= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4b^2} =$$

$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4b^2}.$$

Put $(a+b+c) = 2s$. Then $a+b-c = 2s - 2c$.

$$\therefore h_b{}^2 = \frac{2s \quad 2(s-c)2(s-b)2(s-a)}{4b^2};$$

$$\therefore h_b = \frac{2}{b}[s(s-a)(s-b)(s-c)]^{\frac{1}{2}}.$$

308. (Heron.) *To find the area of a triangle in terms of its sides.*

$$\triangle = \tfrac{1}{2}bh_b = \frac{b}{2}\cdot\frac{2}{b}[s(s-a)(s-b)(s-c)]^{\frac{1}{2}};$$

$$\therefore \triangle = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}.$$

Ex. 291. If $a^2 = b^2 + c^2$, $\therefore \measuredangle A =$ r't $\measuredangle$.
If $a^2 > b^2 + c^2$, $\therefore \measuredangle A >$ r't $\measuredangle$.
If $a^2 < b^2 + c^2$, $\therefore \measuredangle A <$ r't $\measuredangle$.

Ex. 292. Given for the three sides of a triangle numerical expressions in terms of a unit, compute the three altitudes.

Ex. 293. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side. $[a^2 + c^2 - \frac{1}{2}b^2 = 2m_b^2]$.

Ex. 294. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the sect from the foot of that side's altitude to the foot of its median. $\left[j = \frac{a^2 - c^2}{2b}\right]$.

Ex. 295. Given numerical expressions for the sides of a triangle, compute the medians.

$$2m_c = [2(a^2 + b^2) - c^2]^{\frac{1}{2}}.$$

309. Theorem. *The product of two sides of a triangle equals the product of two sects from that vertex making equal angles with the two sides and extending, one to the base, the other to the circle circumscribing the triangle.*

Proof. $\triangle CBD \sim \triangle ABE$.

310. Corollary I to 309. If BD and BE coincide they bisect the angle B;

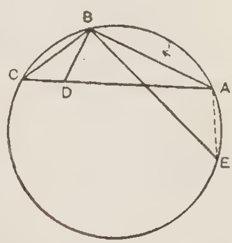

FIG. 120.

$\therefore AB \cdot BC = BD \cdot BE$
$= BD(BD + DE) = BD^2 + BD \cdot DE$
$= \overline{BD}^2 + CD \cdot DA$ (by 248).

Therefore *the square of a bisector together with the product of the sects it makes on a side equals the product of the other two sides.*

311. Corollary II to 309. If BD be an altitude, BE is a diameter, for then $\measuredangle BAE$ is r't; therefore *in any triangle the product of two sides equals the product of the diameter of the circumscribed circle by the altitude upon the third side.*

Ex. 296. To find the bisectors of the angles of a triangle, given the sides. $t_c = \frac{2}{a+b}[abs(s-c)]^{\frac{1}{2}}$.

Ex. 297. To find the radius of the ⊙ circumscribing a triangle. Rule: Divide the product of the three sides by four times the area of the triangle. $R = abc/4\Delta$.

Ex. 298. The in-radius equals area over half sum of sides. $[r = \Delta/s]$.

Ex. 299. The side of an equilateral triangle is

$$b = R(3)^{\frac{1}{2}} = 2r(3)^{\frac{1}{2}}.$$

Ex. 300. The radius of circle circumscribing triangle 7, 15, 20, is $12\frac{1}{2}$. The in-radius is 2.

Ex. 301. To find the radius of an escribed circle. Rule: Divide the area of the triangle by the difference between half the sum of its sides and the tangent side. $[r_1 = \Delta/(s-a)]$.

Ex. 302. $\Delta = (rr_1r_2r_3)^{\frac{1}{2}}$.

Ex. 303. $1/r_1 + 1/r_2 + 1/r_3 = 1/r$.

Ex. 304. The sum of the four squares on the four sides of any quadrilateral is greater than the sum of the squares on the diagonals by four times the square on the sect joining the mid-points of the diagonals.

Ex. 305. The sum of the squares on the four sides of a parallelogram is equal to the sum of the squares on the diagonals.

Ex. 306. The product of the external segments (sects) made on one side by the bisector of an external angle of a triangle equals the square of the bisector together with the product of the other two sides.

Ex. 307. Find r_1, r_2, r_3, when $a=7$, $b=15$, $c=20$.

The Mensuration of the Circle.

312. We assume that with every arc is connected a sect such that if an arc be cut into two arcs, this sect is the sum of their sects; moreover, this sect is not less than the chord of the arc, nor, if the arc be minor, is it greater than the sum of the sects on the tangents from the extremities of the arc to their intersection. This sect we call *the length of the arc*.

313. In practical science, every sect is expressed by the unit sect preceded by a number.

From our knowledge of the number and the unit sect it multiplies, we get knowledge of the sect to be expressed, and we can always construct this expression. For science, the unit sect is the centimeter [$^{cm.}$], which is the hundredth part of the sect called a meter, two marked points on a special bar of platinum at Paris, when the bar is at the temperature of melting ice.

314. If an angle of an equilateral triangle be taken at the center of a circle, the chord it intercepts equals the radius. Therefore the length of a semicircle is not less than three times its radius.

It is in fact greater, since joining a point on the arc of one of these chords to its extremities gives a pair of chords together greater than the radius.

Again, taking any diameter, then the diameter perpendicular to this, then perpendiculars at the

four extremities of these, we have a square of tangents equal to $8r$.

Therefore the length of a semicircle is not greater than four times its radius. It is in fact less, as is seen by drawing a tangent at any fifth point of the circle. The number prefixed to the radius in the expression for the length of the semicircle is designated by the symbol π.

The length of any circle is $2\pi r$.

So the lengths of circles are proportional to their radii.

Historical Note on π.

315. We have proved that π is greater than 3 and less than 4, but the Talmud says: "What is three handbreadths around is one handbreadth through," and our Bible also gives this value 3. [I Kings vii. 23; II Chronicles iv. 2.]

Ahmes (about 1700 B.C.) gave $[4/3]^4 = 3 \cdot 16$. Archimedes placed it between $3\frac{10}{71}$ and $3\frac{1}{7}$. Ptolemy used $3\frac{17}{120}$.

The Hindoo Aryabhatta (b. 476) gave $3 \cdot 1416$; the Arab Alkhovarizmi (flourished 813–833) gave $3 \cdot 1416$; Adriaan Anthoniszoon, father of Adriaan Metius [in 1585] gave $355/113 = 3 \cdot 1415929$; Ludolf van Ceulen [1540–1610] gave the equivalent of over 30 decimal places

$$[\pi = 3 \cdot 141592653589793238462643383279 +]$$

(the decimal fraction was not yet invented), and wished it cut on his tomb at Leyden. Vega gave 140 decimal places; Dase, 200; Richter, 500.

In 1873 Wm. Shanks gave 707 places of decimals.

The symbol π is first used for this number in Jones's "Synopsis Palmariorum Matheseos," London, 1706.

In 1770 Lambert proved π irrational, that is, inexpressible as a fraction. In 1882 Lindemann proved π transcendent, that is, not a root of any algebraic equation with rational coefficients, and hence geometrically inconstructable.

316. Kochansky (1685) gave the following simple construction for the length of the semicircle:

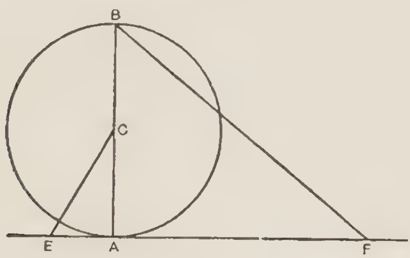

FIG. 121.

At the end-point A of the diameter BA draw the tangent to the circle $\odot C(CA)$. Take $\measuredangle ACE =$ half the angle of an equilateral triangle $= \frac{1}{3}$ r't $\measuredangle$. On the tangent, take $EF = 3r$.

Then BF is with great exactitude the length of the semicircle. In fact $BF = r[13\frac{1}{3} - 2(3)^{\frac{1}{2}}]^{\frac{1}{2}} = 3 \cdot 1415r$

317. Definition. The circle with the unit sect for radius is called the *unit circle.*

318. Definition. The length of the arc of unit circle intercepted by an angle with vertex at center is called the *size* of the angle.

319. Definition. The angle whose size is the unit sect is called a *radian*.

320. Theorem. A radian intercepts on any circle an arc whose length is that circle's radius.

321. Corollary to 320.

If u denote the number of radians in an angle and l the length of the arc it intercepts on circle of radius r, then

$$u = l/r.$$

322. Definition. An arc with the radii to its end-points is called a *sector*.

323. Definition. The area of a sector is the product of the length of its arc by half the radius.

324. Corollary to 323 and 314.

The area of any circle, $\odot = r^2\pi$.

325. Corollary to 324.

The areas of circles are proportional to the areas of squares on their radii.

Ex. 308. The areas of similar polygons are as the squares of corresponding sides.

Ex. 309. Find the length of the circle when $r = 14$ units.

Ex. 310. Find the diameter of a wheel which in a street 19,635 meters long makes 3125 revolutions.

Ex. 311. Find the length and area of a circle when $r = 7$.

Ex. 312. If we call one-ninetieth of a quadrant a *degree of arc* and its angle at the center a *degree of angle*, find the size of this ∡ (size of ∡1°).

Ex. 313. How many degrees in a radian?

Ex. 314. The angles of a triangle are as 1:2:3. Find the size of each. Find the number of degrees in each.

Ex. 315. The angles of a quadrilateral are as 2:3:4:7. Find each in degrees and radians.

Ex. 316. In what regular polygon is every angle $168\frac{1}{2}°$?

Ex. 317. If a r't $\measuredangle$ be divided into $\frac{n}{4}$ congruent parts, how many of them would a radian contain?

Ex. 318. Find the length of the arc pertaining to a central angle of $78°$ when $r = 1 \cdot 5$ meters.

Ex. 319. Find an arc of $112°$ which is 4 meters longer than its radius.

Ex. 320. Calling $\pi = \frac{22}{7}$, find r when $64°$ are $70 \cdot 4$ meters.

Ex. 321. Find the inscribed angle cutting out one-tenth of the circle.

Ex. 322. An angle made by two tangents is the difference between $180°$ and the smaller intercepted arc. Make this statement exact.

Ex. 323. Find the size of half a right angle.

Ex. 324. Find the size of $30°$; $45°$; $60°$.

Ex. 325. How many radians in $\pi°$? in $240°$?

Ex. 326. Express the size of seven-sixteenths of a right angle.

Ex. 327. How many radians in the angle made by the hands of a watch at 5:15 o'clock? at quarter to 8? at 3:30? at 6:05?

Ex. 328. The length of half a quadrant in one circle equals that of two-thirds of a quadrant in another. Find how many radians would be subtended at the center of the first by an arc of it equal in length to the radius of the second.

Ex. 329. Find the number of degrees in an angle whose size is $\frac{1}{2}$; is $\frac{2}{3}$; is $\frac{3}{4}$; is $\frac{11}{21}$; is $\frac{2}{3}\pi$.

Ex. 330. The size of the sum of two angles is $\frac{2 \cdot 5}{12}\pi$ and their difference is $17°$; find the angles.

Ex. 331. How many times is the angle of an isosceles triangle which is half each angle at the base contained in a radian?

Ex. 332. Two wheels with fixed centers roll upon each other, and the size of the angle through which one turns gives the number of degrees through which the other

turns in the same time. In what proportion are the radii of the wheels?

Ex. 333. The length of an arc of 60° is $36\frac{2}{3}$; find the radius.

Ex. 334. Find the circle where $\measuredangle 30°$ is subtended by arc 4 meters long.

Ex. 335. If $\odot$ be area of circle, prove

$$1/[\odot I_1(r_1)]^{\frac{1}{2}} + 1/[\odot I_2(r_2)]^{\frac{1}{2}} + 1/[\odot I_3(r_3)]^{\frac{1}{2}} = 1/[\odot I(r)]^{\frac{1}{2}}.$$

Ex. 336. The perimeters of an equilateral triangle, a square, and a circle are each of them 12 meters. Find the area of each of these figures to the nearest hundredth.

Ex. 337. An equilateral triangle and a regular hexagon have the same perimeter; show that the areas of their inscribed circles are as 4 to 9.

Ex. 338. Find the number of degrees in the arc of a sector whose area equals the square of its radius.

Ex. 339. Find area of sector whose radius equals 25 and the size of whose angle is $\frac{3}{4}$.

Ex. 340. The length of the arc of a sector is 16 meters, the angle is $\frac{1}{8}$ of a r't $\measuredangle$. Find area of sector.

Ex. 341. If 2 $\triangle$s have a common base, their areas are as the segments into which the join of the vertices is divided by the common base.

Ex. 342. The area of a circum-polygon is half perimeter by in-radius [$\frac{1}{2}pr$].

Ex. 343. The area of a rhombus is half the product of its diagonals.

Ex. 344. If we magnify a quad' until a diagonal is tripled, what of its area?

Ex. 345. If the sum of the squares on the three sides of a $\triangle$ = 8 times the square of a median the $\triangle$ is r't-angled.

Ex. 346. Lengthening through A the side b of a $\triangle$ by c and c by b, they become diagonals of a symtra whose area is to that of the $\triangle$ as $(b+c)^2$ to bc.

Ex. 347. If upon the three sides of a r't $\triangle$ as corresponding sides similar polygons are constructed that on the hypothenuse = the sum of those on the $\perp$s.

Ex. 348. The area of any r't △ = the sum of the areas of the two lunes or crescent-shaped figures made by describing semi-⊙s outwardly on the ⊥s and a semi-⊙ on the hypothenuse through the vertex of the r't ∡ [called the lunes of Hippocrates of Chios (about 450 B.C.)].

Ex. 349. (Pappus.) Any two ∥g'ms on two sides of a △ are together = to a ∥g'm on the third side whose consecutive side is = and ∥ to the sect joining the common vertex of the other ∥g'ms to the intersection of their sides ∥ to those of the △ (produced).

Ex. 350. If all the sides of a quad' are unequal, it is impossible to divide it into = △s by straights from a point within to its vertices.

Ex. 351. The joins of the centroid and vertices of a △ trisect it.

Ex. 352. Make a symtra triple a given symtra.

Ex. 353. On each side of a quad' describe a sq' outwardly. Of the four △s made by joining their neighboring corners, two opposite = the other two and = the quad'.

Ex. 354. If from an ∡ α we cut two = △s, one ⫩, the sq' of one of the = sides of the ⫩ △ equals the product of the sides of the other △ on the arms of the ∡ α.

Ex. 355. If any point within a ∥g'm be joined to the four vertices, one pair of △s with ∥ bases = the other.

Ex. 356. One median of a trapezoid cuts it into = parts.

Ex. 357. Transform a given △ into an = ⫩ △.

Ex. 358. Transform a given ⫩ △ into a regular △.

Ex. 359. Construct a polygon ∼ to two given ∼ polygons and = to their sum.

Ex. 360. If a vertex of a △ moves on a ⊥ to the opposite side, the difference of the squares of the other sides is constant.

Ex. 361. The ∡ bi's of a rectangle make a sq', which is half the sq' on the difference of the sides of the rectangle.

Ex. 362. The bisectors of the exterior ∡s of a rectangle make a sq' which is half the sq' of the sum of the sides of the rectangle.

Ex. 363. The sum of the squares made by the bisectors

of the interior and exterior ∡s of a rectangle equals the sq' of its diagonal; their difference is double the rectangle.

Ex. 364. If on the hypothenuse we lay off from each end its consecutive side, the sq' of the mid sect is double the product of the others.

Ex. 365. In $\triangle ABC$, $BD \cdot a = BF \cdot c$.

Ex. 366. In a trapezoid, the sum of the sq's on the diagonals equals the sum of the sq's on the non-∥ sides plus twice the product of the ∥ sides.

Ex. 367. Prove $r_1 + r_2 + r_3 = r + 4R$.

Ex. 368. To bisect a △ by a st' through a given point in a side; by a st' ∥ to a side; ⊥ to a side.

Ex. 369. Trisect a ∔ △ by ∥s.

Ex. 370. A quad' equals a △ with its diagonals and their ∡ as sides and included ∡.

Ex. 371. The areas of △s inscribed in a ⊙ are as the products of their sides.

Ex. 372. Construct an equilateral △, given the altitude.

Ex. 373. △ from ∡s and area.

Ex. 374. Triple the squares of the sides of a △ is quadruple the sq's of the medians.

Ex. 375. Any quad' is divided by its diagonals into four △s whose areas form a proportion.

Ex. 376. $AH \cdot HD = BH \cdot HE$.

Ex. 377. The area of a ∔ r't △ is $\frac{1}{4}c^2$.

Ex. 378. Construct ∔ △ = given △ with same b and h_b.

Ex. 379. Bisect any quad' by a st' from any vertex; from any point in a side.

Ex. 380. Any st' through the bisection-point of a diagonal bisects the ∥gm.

Ex. 381. $3(a^2+b^2+c^2) = 4(m_a{}^2+m_b{}^2+m_c{}^2)$.

Ex. 382. Upon any st' the sum of the ⊥s from the vertices of a △ is thrice the ⊥ from its centroid.

Ex. 383. In r't △, $5c^2 = 4(m_a{}^2+m_b{}^2)$.

Ex. 384. Trisect a quad'.

Ex. 385. Find △ = $\triangle ABC$, but with sides m, n; with side m and adjoining ∡ δ; and opposite ∡ δ.

Ex. 386. Find ∔ $\triangle = \triangle ABC$, but with base m; with side m.

Ex. 387. Find $\triangle$ = given polygon.

Ex. 388. From any point in an equilateral $\triangle$ the three $\perp$s on the sides together = the altitude.

Ex. 389. Sects from the bisection-point of a non-∥ side of a trapezoid to oppsite vertices bisect it.

Ex. 390. If the products of the segments of two intersecting sects are =, their ends are concyclic.

Ex. 391. Area of r't $\triangle$ = product of the segments of the hypothenuse made by $\perp$ from I.

Ex. 392. In r't $\triangle$, areas of $\triangle$s made by h_c are proportional to areas of their in-⊙s.

Ex. 393. $1/h_a + 1/h_b + 1/h_c = 1/r$.

Ex. 394. $h_a h_b h_c = (a+b+c)^3 r^3/abc$.

Ex. 395. If h_a', h_b', h_c' be the perpendiculars from any point within a $\triangle$, upon the sides, prove $h_a'/h_a + h_b'/h_b + h_c'/h_c = 1$.

Ex. 396. $r = \frac{1}{2} AI \cdot BI \cdot CI(a+b+c)/abc$.

Ex. 397. $abc = a(AI)^2 + b(BI)^2 + c(CI)^2$.

Ex. 398. $(AI)^2 + (BI)^2 + (CI)^2 = ab + ac + bc - 6abc/(a + b + c)$.

Ex. 399. $R + r$ = $\perp$s from O on sides.

Ex. 400. In ∔ $\triangle$, if $b = h_b$, then $\frac{5}{8} b = R$.

Ex. 401. $R = 2R$ of $\triangle DEF$.

Ex. 402. Area of $\triangle$ I_1, I_2, $I_3 = abc/2r$.

Ex. 403. If q_a, q_b, q_c be the sides of the 3 sq's inscribed in a $\triangle$, then $1/q_a = 1/h_a + 1/a$; $1/q_b = 1/h_b + 1/b$; $1/q_c = 1/h_c + 1/c$.

Ex. 404. $1/r = 1/h_a + 1/h_b + 1/h_c$; $1/r_1 = -1/h_a + 1/h_b + 1/h_c$; $1/r_2 = 1/h_a - 1/h_b + 1/h_c$; $1/r_3 = 1/h_a + 1/h_b - 1/h_c$.

Ex. 405. $2/h_a = 1/r - 1/r_1 = 1/r_2 + 1/r_3$; $2/h_b = 1/r - 1/r_2 = 1/r_3 + 1/r_1$; $2/h_c = 1/r - 1/r_3 = 1/r_1 + 1/r_2$.

Ex. 406. $h_a/2 = rr_1/(r_1 - r) = r_2 r_3/(r_2 + r_3)$.

Ex. 407. $R^2 = (IO)^2 + 2rR = (I_1O)^2 - 2r_1R$.

Ex. 408. Find the segments of b made by t_b.

Ex. 409. The base of a $\triangle$ is 32 feet and its height 20 feet;

what is the area of the △ formed by drawing a st' ∥ b 5 feet from B? Where must a st' ∥ b be drawn so as to divide the △ into 2 parts of = area?

Ex. 410. Upon each side a of a sq' as diameter semicircles are described within the sq', forming 4 leaves; find the area of a leaf.

CHAPTER XI.

GEOMETRY OF PLANES.

326. Theorem. *Two parallels determine a plane.*

Proof. By definition they are coplanar. Any plane on these parallels would be on three non-costraight points of this given plane, hence (by I 4) identical with it.

327. Corollary to 326. If a plane contains one of two parallels and any point of the other, it contains both parallels.

328. Theorem. *Three planes which do not contain the same straight cannot have more than one point in common.*

Proof. If they had two points in common (by I 5) the straight determined by those two points would be in each.

329. Corollary to 328. If three planes not containing the same straight intersect in pairs, the three straights of intersection [*common sections*, or *meets*] are either copunctal or parallel in pairs.

330. Corollary to 329. If a plane on one of two parallels meet a plane on the other [neither that of the ||s], the meet is parallel to each of the two parallels.

For (by 9) the three planes can have no point

in common, since a point common to the three would be common to the two parallels.

Ex. 411. If a st' cross three copunctal st's, the four are copunctal or coplanar.

Ex. 412. If each of three st's crosses the others, the three are coplanar or copunctal.

Ex 413. The meet of planes determined by two pairs of st's on A is on A.

Ex. 414. If A is on a and α, it is on the intersections of α with planes on a.

331. Problem. *Through a given point A of a given plane α to pass straights a and b in α.*

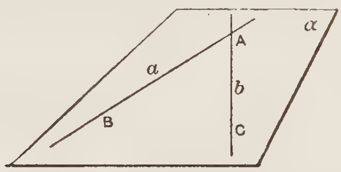

FIG. 122.

Solution. There are (by I 7) in the plane at least three non-costraight points, A, B, C. But (by I 5) A and B determine a straight in the plane α. So do A and C.

332. Problem. *To put two planes on the straight a.*

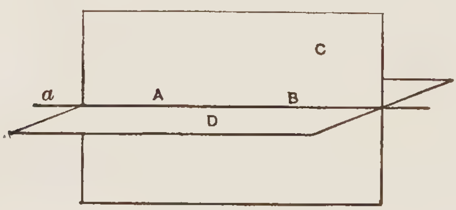

FIG. 123.

Solution. On a (by I 2) are at least two points, A and B. There are (by I 7) at least four non-

costraight non-coplanar points, A, B, C, D. Therefore A, B, and C are not costraight, otherwise (by 11) A, B, C, D would be coplanar. Therefore (by I 3) A, B, C determine a plane which (by I 5) contains a.

Just so A, B, D determine a plane on a.

333. Theorem. *If a straight be perpendicular to each of two intersecting straights, it will be perpendicular to every other straight in their plane and on their point of intersection.*

Hypothesis. Let BP be $\perp$ to BA and BC. Let BD be in plane ABC.

Conclusion. $BP \perp BD$.

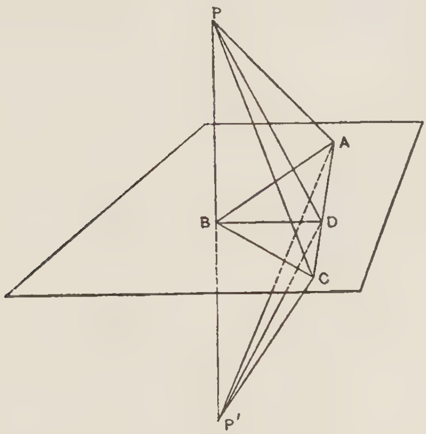

FIG. 124.

Proof. Take A and C on the sides of the angle at B in which BD lies. Let D be the point where AC crosses BD. Take $BP' = BP$. Then (by 43), $\triangle PBA \equiv \triangle P'BA$, and $\triangle PBC \equiv \triangle P'BC$.

$\therefore PA = P'A$ and $PC = P'C$; $\because$ (by 58) $\triangle PAC \equiv \triangle P'AC$.

$\therefore \measuredangle PCD \equiv \measuredangle P'CD$; $\therefore$ (by 43) $\triangle PCD \equiv \triangle P'CD$.

$\therefore PD = P'D$; $\therefore$ (by 58) $\triangle PBD \equiv \triangle P'DB$.

$\therefore \measuredangle PBD \equiv \measuredangle P'BD$; $\therefore BP \perp BD$.

334. Definition. A straight is said to be *perpendicular to a plane* when it is perpendicular to every straight in that plane which passes through its *foot*,—that is, the point it has in common with the plane, called also their *pass*.

Then also the plane is said to be perpendicular to the straight.

335. Definition. A straight is said to be *parallel to a plane* when it has no point in common with the plane.

Then also the plane is said to be parallel to the straight.

336. Definition. A straight neither on the plane, nor parallel nor perpendicular to the plane, is said to be *oblique to the plane*. A sect from a point to a plane, if it be not perpendicular, is called *an oblique*.

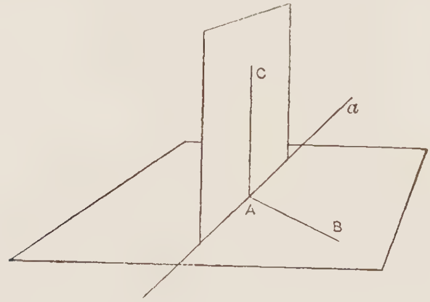

FIG. 125.

337. Problem. *To construct a plane perpendicular to a given straight a at a given point A.*

Solution. Put (by 332) through a two planes. In each of them at A (by 161) erect a perpendicular to a. The plane of these perpendiculars is (by 333 and 334) perpendicular to a at A.

338. Problem. *To construct a plane perpendicular to a given straight a through a given point P not on a.*

Solution. By 160, drop $PA \perp a$.

By 332 and 161, erect another perpendicular AB to a at A. Then (by 333 and 334) plane $PAB \perp a$.

339. Problem. *To erect a perpendicular to a given plane γ at a given point A.*

Solution. Take (by 331) through A two straights, a, b, in the plane γ. Find (by 337) a plane α which at A is $\perp$ to a; also a plane β which at A is $\perp$ b.

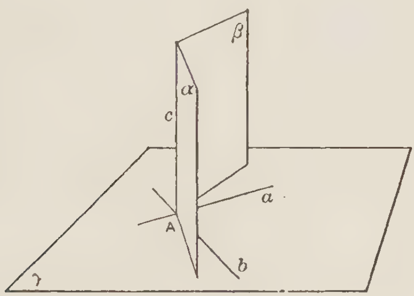

FIG. 126.

These two planes (by 329) intersect in a straight c through A.

Since c is in α, $\therefore$ (by 334) c is $\perp$ a.

Since c is in β, $\therefore$ c is $\perp$ b.

$\therefore$ (by 333 and 334) $c \perp \gamma$.

340. Corollary to 339. Through a given point in a plane there is only one perpendicular to that plane.

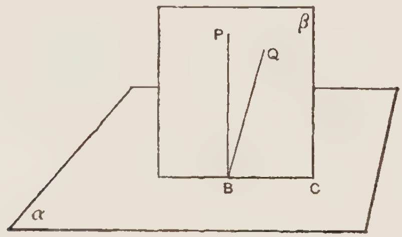

FIG. 127.

Else each would be $\perp$ to BC, the meet of their plane β with the given plane α.

341. Problem. *To drop a perpendicular to a given plane α from a given point P.*

Construction. In α take any straight a.

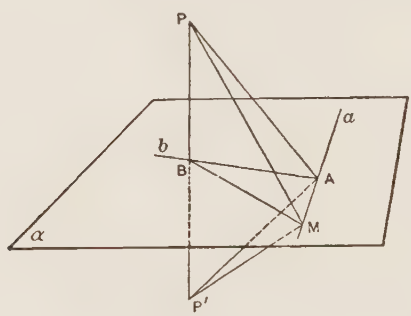

FIG. 128.

From P (by 160) drop $PA \perp a$. In α (by 161) erect $b \perp a$. From P drop $PB \perp b$. Then $PB \perp \alpha$.

Proof. Produce PB through B, taking $BP' = BP$. Then (by 43) $\triangle PBA \equiv \triangle P'BA$; $\therefore PA = P'A$. Further, since $a \perp AP$ and $a \perp AB$, $\therefore$ also (by 333) $a \perp AP'$. Thus if M be a second point on a, $\therefore$ r't $\measuredangle PAM = \measuredangle P'AM$.

$\therefore$ (by 43) $\triangle PAM \equiv \triangle P'AM$; $\therefore PM = P'M$.

$\therefore$ (by 58) $\triangle PBM \equiv \triangle P'BM$; $\therefore \measuredangle PBM$ is r't.

But by construction $\measuredangle PBA$ is r't, $\therefore$ (by 333 and 334) $PB \perp \alpha$.

342. Corollary to 341. From a point P without a plane α, there is only one perpendicular to the plane α.

Take (by 341) $PB \perp \alpha$. Then if A be any other point of α, the r't $\triangle PBA$ has (by 79) the $\measuredangle PAB$ acute.

343. Corollary to 342. From a point to a plane, the perpendicular is less than any oblique. Equal obliques meet the plane in a circle, whose center is the foot of the perpendicular. If through the center of a circle a perpendicular to its plane be taken, then sects from a point of this perpendicular to the circle are equal.

344. Theorem. *If a straight is perpendicular to each of three straights copunctal with it, the three are coplanar.*

Hypothesis. $PB \perp BA$, BD, BC.

Conclusion. BC in plane $BDA[\alpha]$.

Proof. Let plane $PBC[\beta]$ meet plane α in BC'. Then (by 333) $PB \perp BC'$. By hypothesis $PB \perp BC$. But (by 52) in β is only one perpendicular to PB at B. $\therefore BC'$ is identical with BC.

345. Corollary to 344. Through a given point

in a straight there is only one plane perpendicular to that straight.

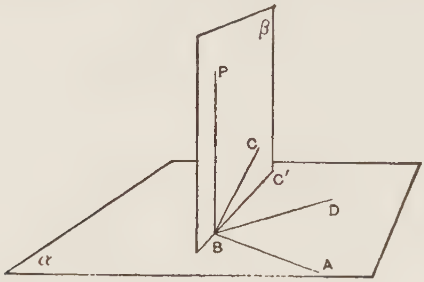

FIG. 129.

346. Theorem. All points, A, which with two fixed points B, C given equal sects, $AB=AC$, are in the plane α bisecting at right angles the sect BC, and inversely every point A' in the plane α bisecting at right angles the sect BC gives $A'B=A'C$.

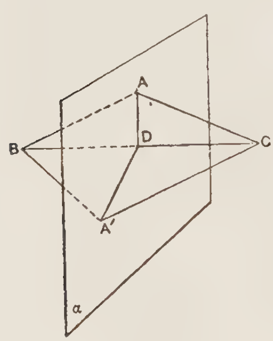

FIG. 130.

Proof. For the straight from A to the bisection point D of BC makes $\triangle ADB \equiv \triangle ADC$ and $\therefore$ (by 344) is in α.

Inversely every straight $A'D$ is $\perp BC$ and $\therefore$ makes $\triangle A'DB \equiv \triangle A'DC$, and $\therefore A'B=A'C$.

347. Corollary to 338. Through a given point without a given straight there is only one plane perpendicular to that straight.

Take (by 338) α through P and $\perp$ to a at B.

Then if A is any other point of a, the r't $\triangle PBA$ has (by 79) the $\measuredangle PAB$ acute. So plane β through P and $\perp$ to a could not pass through A, and so, passing through B, it is (by 345) identical with α.

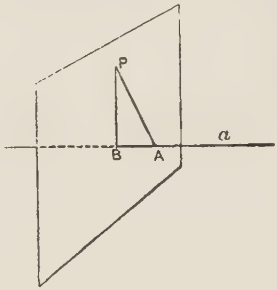

Fig. 131.

348. Theorem. If $PB \perp BAC$ and $BA \perp AC$, then $PA \perp AC$.

Proof. Make $AC = BP$. $\therefore$ (by 43) $\triangle CAB \equiv \triangle PBA$. $\therefore CB = PA$, $\therefore$ (by 58) $\triangle CBP \equiv \triangle PAC$, $\therefore \measuredangle CBP \equiv \measuredangle PAC$.

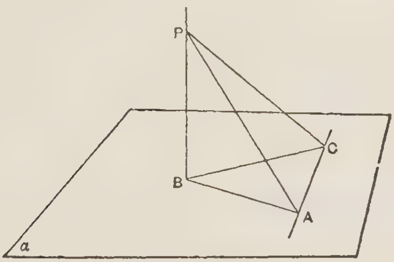

Fig. 132.

But by hypothesis $\measuredangle CBP$ is right.

Ex. 415. If $PB \perp BAC$ and $PA \perp AC$, then $BA \perp AC$.

Ex. 416. If $PB \perp BAC$ and $BA \perp AC$, all $\perp$s to AC from points in PB go to A.

Ex. 417. If $PH \perp ABC$ (H is orthocenter), then $PA \perp$ to $AK \parallel BC$.

349. Theorem. *Two perpendiculars to a plane are coplanar.*

Hypothesis. PB, $P'A \perp \alpha$ at B, A.

Conclusion. P' is in the plane ABP.

Proof. In α erect $AC \perp AB$.

$\therefore$ (by 348) $AC \perp AP$. But by hypothesis $AC \perp AP'$.

$\therefore$ (by 344) A, P', P, B are coplanar.

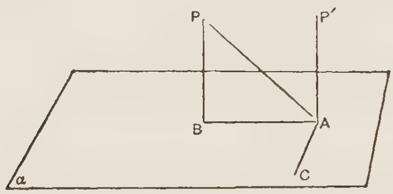

FIG. 133.

350. Corollary to 349 and 342. Two perpendiculars to a plane are parallel.

351. Inverse of 350. If the first of two parallels is perpendicular to a plane, the second is also perpendicular to that plane.

For the perpendicular erected to the plane from the foot of the second is (by 350) parallel to the first and so (by IV) identical with the second.

352. Theorem. *If one plane be perpendicular to one of two intersecting straights, and a second plane perpendicular to the second, they meet and their meet is perpendicular to the plane of the two straights.*

Hypothesis. Let α be $\perp CA$ at A and β be $\perp CB$ at B.

Proof. The meet AD of α with plane ACB is (by 334) $\perp AC$, and likewise $BD \perp BC$; $\therefore$ (by 77)

AD meets BD. Thus the planes α and β, having D in common, meet in DP, and (by 351) their meet PD is $\perp$ to a straight through $D \parallel$ to AC, and also $\perp$ to a straight through $D \parallel$ to BC.

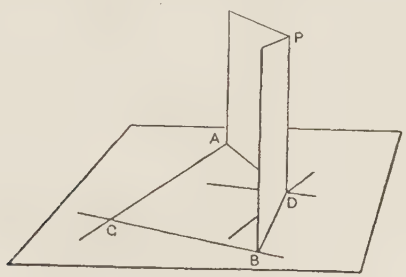

FIG. 134.

353. Theorem. *Two straights, each parallel to the same straight, are parallel to one another, even though the three be not coplanar.*

For a plane $\perp$ to the third will (by 351) be $\perp$ to each of the others; $\therefore$ (by 350) they are $\parallel$.

Ex. 418. Are st's $\parallel$ to the same plane $\parallel$? Are planes $\parallel$ to the same st' $\parallel$?

Ex. 419. A plane $\parallel$ to the meet of two planes meets them in $\parallel$s.

354. Definition. The *projection of a point* upon a plane is the foot of the perpendicular from the point to the plane.

The *projection of a straight* upon a plane is the assemblage of the projections of all points of the straight.

355. Theorem. *The projection of a straight on a plane is the straight through the projections of any two of its points.*

Given A', P', B', the projections of A, P, B, points of the straight AB, on the plane α.

To prove P' in the straight $A'B'$.

Proof. A, A', B, B' are (by 349) coplanar. P is (by I 5) in this same plane, $\therefore$ (by 350 and 327) so is PP'; $\therefore$ (by 9) A', P', B' are costraight.

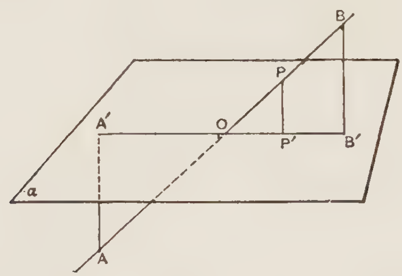

FIG. 135.

356. Corollary to 355. A straight and its projection on a given plane are coplanar. If a straight intersects a plane, its projection passes through the point of intersection. A straight parallel to a plane is parallel to its projection on that plane.

357. Theorem. *A straight makes with its own*

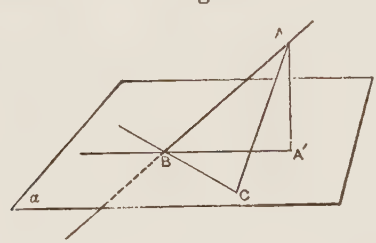

FIG. 136.

projection upon a plane a less angle than with any other straight in the plane.

Hypothesis. Let A' and BA' be the projection of A and BA on α, and BC any other straight in α, through B.

Conclusion. $\measuredangle ABA' < \measuredangle ABC$.

Proof. Take $BC = BA'$. Then $AA' < AC$. (The perpendicular is the least sect from a point to a straight.)

$\therefore \measuredangle ABA' < \measuredangle ABC$.

(In two $\triangle$'s, if $a = a'$, $b = b'$, $c < c'$, then $\measuredangle C < \measuredangle C'$.)

358. Definition. The angle between a straight and its projection on a plane is called the *inclination* of the straight to the plane.

Ex. 420. One of three copunctal st's makes = $\measuredangle$s with the others if its projection on their plane bisects their $\measuredangle$.

Ex. 421. An oblique makes with some st' in the plane through its foot any given $\measuredangle <$ the supplement of its inclination and $>$ its inclination.

Ex. 422. Equal obliques from a point to a plane are equally inclined to it.

359. Definition. *Parallel planes* are such as nowhere meet.

360. Theorem. *Planes perpendicular to the same straight are parallel.*

Proof. They cannot (by 345 and 347) have a point in common.

Ex. 423. A st' and a plane $\perp$ to the same st' are $\parallel$.

361. Theorem. *Every plane through only one of two parallels is parallel to the other.*

Given $AB \parallel CD$ in α, and β another plane through AB.

To prove $CD \parallel \beta$.

Since AB is in α and in β, it contains (by 9) every point common to the two planes.

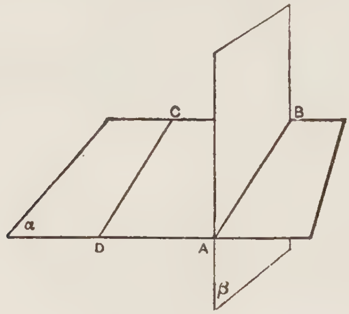

FIG. 137.

But CD is wholly in α. So to meet β it must have a point in common with α and β, that is, it must meet AB. But by hypothesis $AB \parallel CD$.

Ex. 424. Through a given point to draw a st' ∥ to two given planes.

Ex. 425. If $a \parallel \alpha$, and b the meet of α with β, β on a then $a \parallel b$.

Ex. 426. Through A determine a to cut b and c.

362. Problem. *Through either of two straights not coplanar to pass a plane parallel to the other.*

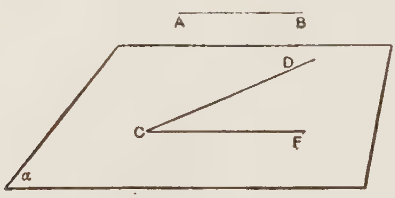

FIG. 138.

If AB and CD are the given straights, take $CF \parallel AB$. Then (by 361) $DCF \parallel AB$.

Determination. There is only one such plane. For, through DC any plane $\parallel AB$ meets plane ABC in the parallel to AB through C, $\therefore$ is identical with CDF.

Ex. 427. Through a point without a plane pass any number of st's $\parallel$ to that plane.

Ex. 428. Through a point without a st' pass any number of planes $\parallel$ to that st'.

Ex. 429. Planes on $a \parallel \alpha$ meet α in $\parallel$s.

Ex. 430. If $a \parallel \alpha$ and $a \parallel \beta$, then $a \parallel$ to the meet $\alpha\beta$.

363. Problem. Through any given point P to pass a plane parallel to any two given straights,

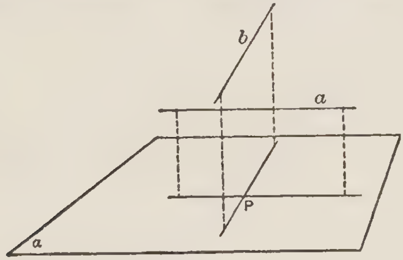

FIG. 139.

a, b. [The plane determined by the parallel to a through P, and the parallel to b through P.] [There is only one such plane.]

Ex. 431. Through two non-coplanar straights one and only one pair of $\parallel$ planes can be passed.

364. Theorem. *The intersections of two parallel planes with a third plane are parallel.*

Proof. They cannot meet, being in two parallel planes; yet they are coplanar, being in the third plane.

365. Corollary to 364. Parallel sects included between parallel planes are equal.

Ex. 432. If two ‖ planes meet two ‖ planes, the four meets are ‖.

Ex. 433. If a ‖ to the meet $\alpha\beta$, then $a \parallel \alpha$ and $a \parallel \beta$.

Ex. 434. If $\alpha \parallel \beta$, $a \perp \alpha$ is $\perp \beta$.

Ex. 435. Through A draw $\alpha \parallel \beta$. [Solution unique.]

Ex. 436. If, in α, a cross a', in β, b cross b', and $a \parallel b$, $a' \parallel b'$, then $\alpha \parallel \beta$.

Ex. 437. Through A all st's ‖ α are coplanar.

Ex. 438. Two planes ‖ to a third are ‖.

Ex. 439. The intercepts on ‖s between α and $a \parallel \alpha$ are =.

Ex. 440. If $AB \parallel \alpha$ and $BC \parallel \alpha$, then plane $ABC \parallel \alpha$.

Ex. 441. If three sects are = and ‖, the △s of their adjoining ends are ≡ and ‖.

Ex. 442. If A in $\alpha \parallel a$, $AB \parallel a$ is in α.

366. Theorem. *If two angles have their sides respectively parallel and on the same side of the straight through their vertices, they are equal.*

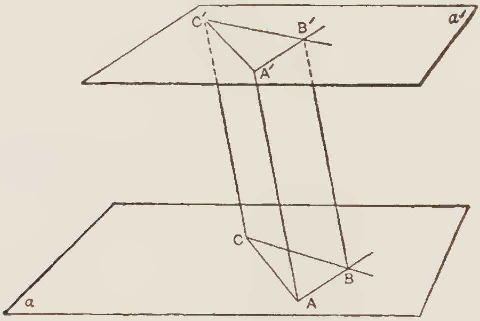

Fig. 140.

Hypothesis. $AB \parallel A'B'$ with A and A' on the same side of BB'; also $CB \parallel C'B'$ with C and C' on same side of BB'.

Proof. From A take (by 66) $AA' \parallel BB'$; $\therefore$ (by 95) $AA' = BB'$ and $A'B' = AB$. In same way $CC' \parallel$ and $= BB'$ and $B'C' = BC$. But then $AA' = CC'$ and (by 353) $AA' \parallel CC'$; $\therefore$ (by 100) $AC = A'C'$; $\therefore$ (by 58) $\triangle ABC \equiv \triangle A'B'C'$.

367. Corollary to 366. Parallels intersecting the same plane are equally inclined to it.

368. Definition. Let two planes, α, β, intersect in the straight a. Let A and A' be points on a. Erect now at A and A' perpendiculars to a in one hemiplane α' of α, and also in hemiplane β' of β. Then (by 366) the angle of the perpendiculars at A is equal to the angle of the perpendiculars at A'.

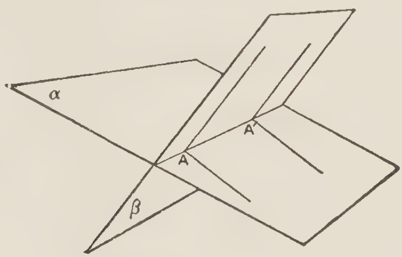

Fig. 141.

We call this angle *the inclination of the two hemiplanes* α' and β'.

When the inclination is a right angle the planes are said to be *perpendicular* to each other.

369. Theorem. *If a straight is perpendicular to a given plane, any plane containing this straight is perpendicular to the given plane.*

Proof. At the foot of the given perpendicular erect in the given plane a perpendicular to the

meet of the planes. From the definition of a perpendicular to a plane (334) the given perpendicular makes with this a r't $\measuredangle$; but this angle is (by 368) the inclination of the planes.

Ex. 443. A plane $\perp$ the meet of two planes is $\perp$ to each; and inversely.

Ex. 444. Through a in α draw $\beta \perp \alpha$.

369 (b). Corollary to 369. A straight and its projection on α determine a plane perpendicular to α.

370. Theorem. *If two planes are perpendicular to each other, any straight in one, perpendicular to their meet, is perpendicular to the other.*

371. Corollary to 370. If two planes are perpendicular to each other, a straight from any point in their meet, perpendicular to either, lies in the other.

For the perpendicular to their meet in one is perpendicular to the other, and (by 340) there is only one perpendicular to a plane at a point.

Ex. 445. If A in $\alpha \perp \beta$, from A, $a \perp \beta$ is in α.

Ex. 446. If a st' be $\parallel$ to a plane, a plane $\perp$ to the st' is $\perp$ to the plane.

372. Corollary to 371. If each of two intersecting planes is perpendicular to a given plane, their meet is perpendicular to that plane.

Proof. The perpendicular to this third plane from the foot of the meet of the others is (by 371) in both of them.

Ex. 447. Through a st' $\parallel$ α to pass $\beta \parallel \alpha$.

Ex. 448. Through a draw $\alpha \perp \beta$.

Ex. 449. Through A draw $\alpha \perp \beta$ and γ.

373. Theorem. *If two straights be cut by three parallel planes the corresponding sects are proportional.*

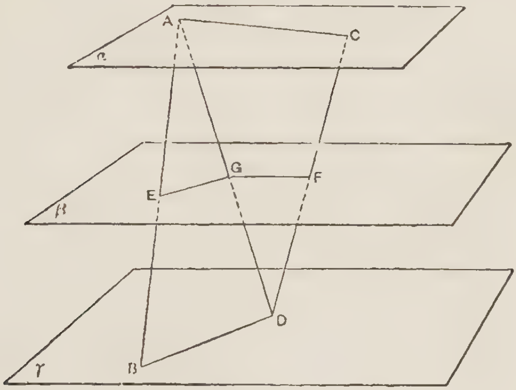

FIG. 142.

Let AB, CD be cut by the parallel planes α, β, γ in A, E, B and C, F, D.

To prove $AE:EB = CF:FD$.

Proof. If AD cut β in G, then (by 364) $EG \parallel BD$ and $AC \parallel GF$.

$\therefore$ (by 235) $AE:EB = AG:GD$ and $AG:GD = CF:FD$.

$\therefore AE:EB = CF:FD$.

Ex. 450. Investigate the inverse of 373.

374. Theorem. *Two straights not coplanar have one, and only one, common perpendicular.*

Given a and b not coplanar.

To prove there is one, and only one, straight perpendicular to both.

Proof. Through any point A of a take $c \parallel b$. Then (by 361) the plane ac or $\alpha \parallel b$. The projection

b' of b on α cuts a, say in B'; else were $a \parallel b' \parallel b$. Then, in plane $b'b$, $B'B$ drawn $\perp b'$ is (by 370) $\perp a$

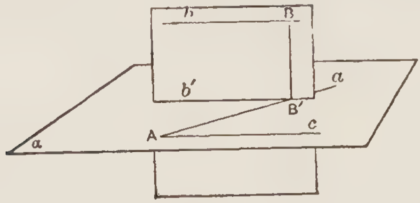

FIG. 143.

and (by 74) also $\perp b$, and is the only common perpendicular to a and b. For any common perpendicular meeting a at B'' is $\perp b''$ through $B'' \parallel b$, which is in α, and $\therefore \perp \alpha$, hence B'' is a point of b' the projection of b on α; $\therefore$ identical with B', the cross of b' with a.

375. Corollary to 374. Their common perpendicular is the smallest sect between two straights not coplanar.

For (by 142) $BB' < BA$.

Ex. 451. No st's joining points in two non-coplanar st's can be $\parallel$.

Ex. 452. From A, B, C, costraight, are dropped to a non-coplanar st' $\perp$s AD, BE, CF. Prove $AB:BC = DE:EF$.

Ex. 453. A plane $\perp$ to the common $\perp$ to two st's at its bisection-point, bisects every sect from one st' to the other.

Ex. 454. *Principle of Duality.* When any figure is given we may construct a *dual* figure by taking planes instead of points, and points instead of planes, but straights where we had straights.

The figure dual to four non-coplanar points is four non-copunctal planes. State the dual of the following:

Two planes determine a straight.

Three non-costraight planes determine a point.

A straight and a plane not on it determine a point.

Two straights through a point determine a plane.

Ex. 455. A polygon whose vertices are always necessarily coplanar is what?

Ex. 456. From each point in a plane costraight with and equally inclined to two others the two $\perp$s to them are $=$.

Ex. 457. If two planes are respectively $\perp$ to two others and the intersection of the first pair $\parallel$ to that of the second pair, the inclinations are $=$ or supplemental.

Ex. 458. Parallels have as projections on any plane parallels or points.

Ex. 459. Parallel sects are proportional to their projections on a plane.

Ex. 460. From a point, $\perp$s to two planes make an $\measuredangle$ $=$ or supplemental to the inclination of the planes.

Ex. 461. A st' has the same inclination to $\parallel$ planes.

Ex. 462. If three meets of three planes are $\parallel$, the sum of the three inclinations is two r't $\measuredangle$s.

Ex. 463. If a st' is $\parallel$ to each of two planes, any plane on it cuts them in $\parallel$s.

Ex. 464. Through a point without two non-coplanar st's passes, in general, a sin le st' cutting both.

Ex. 465. Why does each foot of a three-legged stool meet the floor while one foot of a four-legged chair may be above the floor?

Ex. 466. If a, b non-coplanar, $\alpha \perp a$ meets $\beta \perp b$ in $c \perp \gamma \parallel a$ and b.

Ex. 467. If two projections of a trio of points on two intersecting planes give costraight trios, the original three are costraight. State the exception.

Ex. 468. No oblique to a plane makes equal angles with three straights in the plane.

Ex. 469. Draw a plane with the same inclination to two given planes.

Ex. 470. Draw a straight that shall cross three straights, no two coplanar.

Ex. 471. Draw a st' to cross two given non-coplanar st's and || to another given st'. [Show solution, in general, unique.]

Ex. 472. In a plane find a point which joined with three given points without the plane gives equal sects.

Ex. 473. The projection on α of r't $\measuredangle (a, b)$ is a r't $\measuredangle$ if $a \parallel \alpha$.

Ex. 474. If a perpendicular to a plane be projected on any second plane this projection is at right angles to the meet or intersection of the planes.

Ex. 475. From a point without a plane, if there be drawn the perpendicular to the plane and also a perpendicular to a straight in the plane, the join of the feet of these perpendiculars is at right angles to the straight.

Ex. 476. Two planes being given perpendicular to each other, draw a third perpendicular to both.

Ex. 477. Three planes, no two parallel, either intersect in one point (are copunctal) or in one straight (are co-straight) or have their three intersection-straights (meets) parallel.

Ex. 478. If two straights be at right angles either is in the plane through their point of intersection (cross) perpendicular to the other.

Ex. 479. If three planes have two of their intersection-straights parallel, the third is parallel to both.

Ex. 480. All straights on two intersecting straights, but not on their cross, are coplanar.

Ex. 481. If the vertices of a triangle give equal sects when joined to a point without their plane, the foot of the perpendicular from this point to the plane is the triangle's circumcenter.

Ex. 482. All points which joined to three given points give three equal sects are where?

Ex. 483. All coplanar points which joined to a given point give equal sects are where?

Ex. 484. If a plane contains one straight perpendicular to a second plane, every straight in the first plane per-

pendicular to the intersection-straight (the meet) of the planes is also perpendicular to the second.

Ex. 485. Any plane is equally inclined to two parallel planes.

Ex. 486. If $\perp$s from a point to two intersecting planes be = it determines with their meet a plane equally inclined to them.

Ex. 487. Construct a plane containing a given straight and perpendicular to a given plane.

Ex. 488. Two perpendiculars from a point to two intersecting planes determine a plane perpendicular to the meet of the two planes.

Ex. 489. If each of three planes be perpendicular to the other two, their three meets are also perpendicular to the planes and to one another.

Ex. 490. If any number of planes perpendicular to a given plane have a common point, they have a common meet (intersection-straight).

Ex. 491. If the meets of several planes are parallel, the perpendiculars to them from any given point are coplanar.

Ex. 492. Perpendiculars from two vertices of a parallelogram to a plane through the other two are equal.

Ex. 493. Two sides of an equilateral triangle are equally inclined to any plane through the third.

Ex. 494. If two straights be not coplanar, find a point in one which, joined to two given points in the other, gives equal sects.

Ex. 495. If $\alpha \perp \beta$ and γ. and the meet $\alpha\beta \parallel \alpha\gamma$, then $\beta \parallel \gamma$.

Ex. 496. If the four sides of a quadrilateral be not coplanar it is called *skew*. No three sides of a skew quadrilateral are coplanar, nor can its four $\measuredangle$s be r't.

Ex. 497. If a sect divide one pair of opposite sides of a skew quadrilateral proportionally, and another divide the other pair in another proportion, these two sects will cross and each cut the other as it cuts the sides.

Ex. 498. Find that point in a given plane from which the sum of the sects to two given points on the same side is least.

Ex. 499. If A and a are in α, and B not, find the point in a from which the sum of sects to A and B is least.

Ex. 500. Every plane not on a vertex cuts an even number of the sides of a skew quad' internally and an even number externally.

Ex. 501. If non-coplanar $\triangle$s ABC, $A'B'C'$ have AA', BB', CC' copunctal, then the three pairs of sides AB, $A'B'$; AC, $A'C'$; BC, $B'C'$ intersect in three costraight points.

Ex. 502. If two st's in one plane be equally inclined to another plane they make = ∡s with the common section of these planes.

Ex. 503. If three planes be each $\perp$ to the other two, the sq' of the sect from their common intersection to another point equals the sum of the sq's of the three $\perp$s from that point to the planes.

Ex. 504. If three st's be each $\perp$ to the other two, twice the sq' of the sect from their common intersection to another point equals the sum of the sq's of the three $\perp$s from that point to the st's.

Ex. 505. Draw a st' to cut three given non-intersecting st's so that the intercepts may be as two given sects.

Ex. 506. If a plane cut a tetrahedron in a ||g'm, the plane is || to two opposite edges.

Ex. 507. The aggregate of all points is divided by four planes into (in general) fifteen regions.

Ex. 508. The medians of a skew quadrilateral bisect one another.

Ex. 509. If two medians of a skew quadrilateral be $\perp$ the diagonals are =, and sections || to them are ||g'ms of = perimeter.

CHAPTER XII.

POLYHEDRONS AND VOLUMES.

Polyhedrons.

376. Definition. A *tetrahedron* is the figure constituted by four non-coplanar points, their sects and triangles.

The four points are called its *summits*, the six sects its *edges*, the four triangles its *faces*. Every summit is said to be *opposite* to the face made by the other three; every edge opposite to that made by the two remaining summits.

A point is *within* the tetrahedron if it is within any sect made by any summit and a point within its opposite face. Points not within or *on* are *without*.

The faces taken together are called the *surface* of the tetrahedron.

377. A *polyhedron* is the figure formed by n plane polygons such that each side is common to two.

The polygons are called its faces, and taken together, its surface. Their sects are its edges; their vertices its summits.

A convex polyhedron is one through no edge of which pass more than two faces, and which has no summits on different sides of the plane of a face.

A polyhedron of five, six, eight, twelve, twenty faces is called a pentahedron, hexahedron, octahedron, dodecahedron, icosahedron.

378. A *pyramid* is a polyhedron of which all the faces, except one, are copunctal. This one face is called the *base*, and the summit not on it the *apex*.

The faces which meet at the apex are called *lateral* faces, and together the lateral surface; the edges meeting at the apex are called lateral edges.

The perpendicular from the apex to the plane of the base is called the *altitude* of the pyramid.

379. Euler's Theorem. *In any convex polyhedron the number of faces increased by the number of summits exceeds by two the number of edges.*

To prove $F+S=E+2$.

Proof. Let e be any edge joining the summits A, B and the faces α, β; and let e vanish by the approach of B to A. If α and β are neither of them triangles, they both remain, though reduced in rank and no longer collateral, and the polyhedron has lost one edge e and one summit B.

If β is a triangle and α no triangle, β vanishes with e into an edge through A, but α remains. The polyhedron has lost two edges of β, one face β, and one summit B.

If β and α are both triangles, β and α both vanish with e, five edges forming those triangles are reduced to two through A, and the polyhedron has lost three edges, two faces, and the summit B.

In any one of these cases, whether one edge and one summit vanish, or two edges disappear with

a face and a summit, or three edges with a summit and two faces, the truth or falsehood of the equation

$$F+S=E+2$$

remains unaltered.

By causing all the edges which do not meet any face to vanish, we reduce the polyhedron to a pyramid upon that face. Now the relation is true of the pyramid; therefore it is true of the undiminished polyhedron.

380. Theorem. *The sum of the face angles of any convex polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has summits.*

To prove $\Sigma=(S-2)4$ r't $\measuredangle$.

Proof. Since E denotes the number of edges, $2E$ is the number of sides of the faces.

Taking an exterior angle at each vertex, the sum of the interior and exterior angles is $2E2$ r't $\measuredangle$, or $E4$ r't $\measuredangle$. But the exterior angles of each face make 4 r't $\measuredangle$; $\therefore$ the exterior angles of F faces make $F4$ r't $\measuredangle$.

$$\therefore \Sigma=E4 \text{ r't } \measuredangle - F4 \text{ r't } \measuredangle = (E-F)4 \text{ r't } \measuredangle.$$

But (by 379) $\quad F+S=E+2;$

$$\therefore E-F=S-2;$$

$$\therefore \Sigma=(S-2)4 \text{ r't } \measuredangle.$$

Ex. 510. The number of face angles in the surface of any polyhedron is twice the number of its edges.

Ex. 511. If a polyhedron has for faces only polygons

with an odd number of sides, it must have an even number of faces.

Ex. 512. If the faces of a polyhedron are partly of an even, partly of an odd, number of sides, there must be an even number of odd-sided faces.

Ex. 513. The number of face angles on a polyhedron can never be less than thrice the number of faces.

Ex. 514. In every polyhedron $\frac{3}{2}S \overset{=}{<} E$.

Ex. 515. In any polyhedron $E+6 \overset{=}{<} 3S$.

Ex. 516. In any polyhedron $E+6 \overset{=}{<} 3F$.

Ex. 517. In every polyhedron $E<3S$.

Ex. 518. In every polyhedron $E<3F$.

Ex. 519. In a polyhedron, not all the summits are more than five sided; nor have all the faces more than five sides.

Ex. 520. There is no seven-edged polyhedron.

Ex. 521. For every convex polyhedron the sum of the face angles is four times as many right angles as the difference between the number of edges and faces.

Ex. 522. How many regular convex polyhedrons are possible?

Ex. 523. In no polyhedron can triangles and three-faced summits both be absent; together are present at least eight.

Ex. 524. A polyhedron without triangular and quadrangular faces has at least twelve pentagons; a polyhedron without three-faced and four-faced summits has at least twelve five-faced.

The volume of tetrahedrons and polyhedrons.

381. Theorem. *The product of an altitude of a tetrahedron by the area of its base is independent of what summit one chooses as apex.*

Proof. From H and H', feet of altitudes from D and C, drop perpendiculars HK, and $H'K'$ to AB.

Then $KD \perp AB$. [If two planes ABC and HKD are at r't ∡'s, then a st' AK in one $\perp$ to their

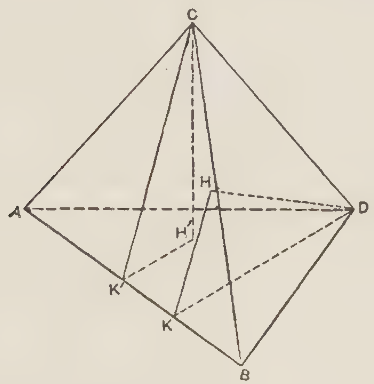

FIG. 144.

intersection KH is also $\perp$ to the other HDK and $\therefore AK \perp KD$.]

In same way $AK' \perp K'C$.

$\therefore$ (by 366) $\measuredangle K \equiv \measuredangle K'$.

$\therefore$ the r't triangles HKD and $H'K'C$ are similar.

$\therefore DK : CK' = DH : CH'$.

$\therefore DK \cdot CH' = CK' \cdot DH$.

$\therefore \frac{1}{2}AB \cdot DK \cdot CH' = \frac{1}{2}AB \cdot CK' \cdot DH$.

382. Definition. One-third the product of base and altitude of a tetrahedron T is called the *volume of tetrahedron* T and designated by $V(T)$.

383. Convention. A plane through an edge of a tetrahedron and a point of the opposite edge is called a *transversal plane;* this cuts the tetrahedron into two tetrahedra with common altitude whose bases are coplanar. Such a partition is called a *transversal partition of the tetrahedron.*

384. Theorem. *The volume of any tetrahedron is equal to the sum of the volumes of all the tetrahedra which arise from the first by making successively a set of transversal partitions.*

Proof. From the distributive law in our sect-calculus follows immediately that the volume of any tetrahedron is equal to the sum of the volumes of two tetrahedra which come from the first by any transversal partition. Thus if the given tetrahedron $ABCD$ is cut by the transversal plane ACE passing through the edge AC, the two tetrahedra so obtained, $AEBC$ and $AEDC$, have in common the altitude h_c from C. Moreover, the area of the triangle ABD is equal to the sum of the areas of AEB and AED.

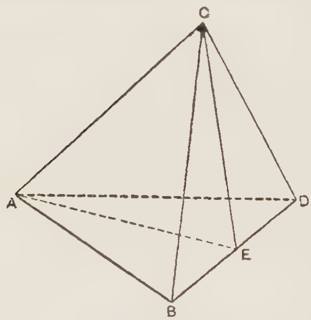

FIG. 145.

Thus $V(T_1) + V(T_2) = \frac{1}{3}h_c \cdot A(\triangle_1) + \frac{1}{3}h_c \cdot A(\triangle_2) = \frac{1}{3}h_c [A(\triangle_1) + A(\triangle_2)] = \frac{1}{3}h_c \cdot A(\triangle) = V(T)$.

Now our theorem follows merely by repeated application of this single result.

385. Theorem. *However a tetrahedron is cut by a plane, this partition can be obtained in a set of transversal partitions using not more than two other planes.**

Proof. Passing the case in which the plane

* See G. Veronese, Atti del R. Istituto Veneto, t. vi, s. vii; 1894–95.

itself goes through an edge of the tetrahedron *ABCD*, there remain three cases:

I. The plane passes through a single summit, for example, *A*, thus cutting the tetrahedron *ABCD* into the tetrahedron *ABXY* and the pyramid with quadrilateral base *AXYCD*.

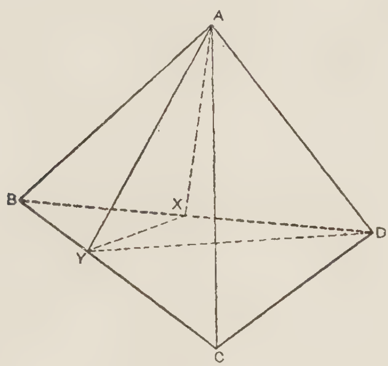

FIG. 146.

But this partition is in the set of transversal partitions obtained by taking successively the planes *ADY* and *AYX*.

II. The plane cuts the three edges copunctal in a summit, for example, *A*, thus cutting the tetrahedron *ABCD* into the tetrahedron *AXYZ* and the convex polyhedron *XYZBCD*. But this partition is in the set of transversal partitions obtained by taking successively the planes *BDY*, *BYZ*, *YZX*.

III. The plane cuts two pairs of opposite edges, for example, *AB*, *CD*, and *AC*, *BD*.

Thus the tetrahedron is cut into the two poly-

hedrons $ADWXYZ$ and $BCWXYZ$. Draw the two planes BDX and ACZ. By these at the same time the polyhedron $ADWXYZ$ is cut into the

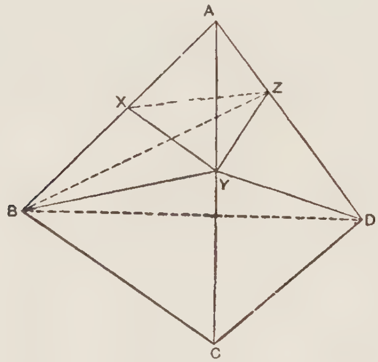

FIG. 147.

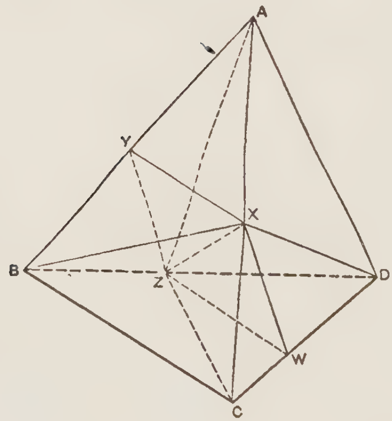

FIG. 148.

three tetrahedra $XZAY$, $XZAD$, $XZDW$, and the polyhedron $BCWXYZ$ cut into the three tetrahedra $XZBY$, $XZBC$, $XZCW$.

But that these six tetrahedra form a set obtainable by transversal partitions is seen by taking first the plane BDX, and then in the tetrahedron $BDXC$ successively the planes CXZ and ZXW; in the tetrahedron $BDXA$ successively the planes AXZ and YZX.

386. Theorem. *If a tetrahedron T is in any way cut into a certain finite number of tetrahedra T_k, then is always the volume of the tetrahedron T equal to the sum of the volumes of all the tetrahedra T_k.*

Proof. The plane α of any one face of any one of the tetrahedra T_k, say, T_1, makes in T a partition which, by 385, can be obtained in a set of transversal partitions of T made by introducing two other planes β, γ, cutting T into T_t. When α cuts any tetrahedron T_k, say T_2, add in this T_2 the two planes δ, ϵ, making the corresponding set of transversal partitions in this T_2.

When β cuts one of these new tetrahedra, say T_{2a}, add in it the requisite two planes ζ, η. So do for any tetrahedron T_3 met by β. Then in the same way successively for γ.

Now produce a second face of T_1, say θ, to cut those tetrahedra T_t in which this face is situated. Add in each of these T_t the two planes to make the transversal partition. When any plane cuts a tetrahedron already existing, add the requisite two planes.

Now produce a third face of T_1 to the nearest tetrahedron transversally obtained from T. Finally take in the fourth face of T_1. Then T_1 appears

in a set of transversal partitions of T; while if cut itself, it is by a set of transversal partitions of T_1. In the same way for every other T_k.

Thus after a finite number of constructions we reach by a set of transversal partitions of T a final set of tetrahedra T_j, which are at the same time also reached by transversal partitions of the tetrahedra T_k.

386*b*. Of the fundamental theorem 386:

For any partition of a tetrahedron into tetrahedra the sum of their volumes equals its volume, the following alternative proof is due to S. O. Schatunovsky of Odessa.

386*c*. Partition method I. Cut a face, for example BCD (Fig. 149), of the tetrahedron in

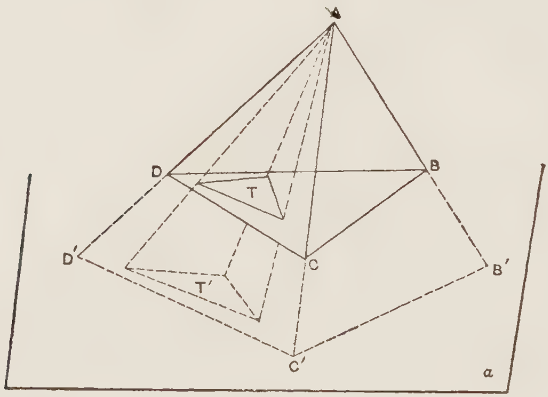

FIG. 149.

question $ABCD$ into a finite number n of triangles, and join their vertices with the summit A. The tetrahedra so obtained have a common summit A,

and the faces opposite it coplanar. [For the sake of distinctness, Fig. 149 shows only *one* such part-tetrahedron.]

By 280, the area of the base of the original tetrahedron equals the sum of the areas of all the bases of the part-tetrahedra. It and they have the same altitude. Our multiplication of sects is distributive. Therefore its volume equals the sum of theirs.

386*d*. This I contains transversal partition (383) as a special case [i.e. for $n=2$].

386*e*. Partition method II. Cut the tetrahedron $ABCD$ so that all summits of the part-tetrahedra lie on three edges meeting in the same apex, for example on the edges AB, AC, AD (Fig. 150).

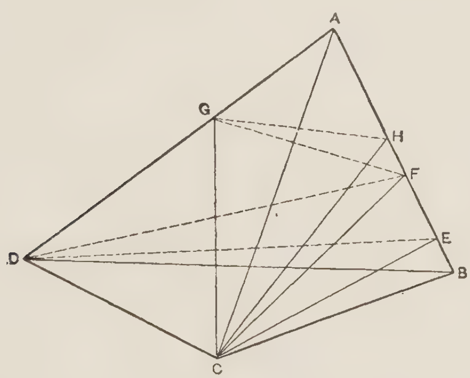

FIG. 150.

Then on the base BCD are, besides the points B, C, D, no summits of the part-tetrahedra; and consequently the face BCD is a face of a part-tetrahedron. The fourth summit E of this part-tetrahedron may lie on the edge AB (Fig. 150).

Therefore this partition into n part-tetrahedra may be obtained by first cutting the tetrahedron $ABCD$ by a transversal partition (here through DC) into two tetrahedra $BCDE$ and $ECDA$, and then $ECDA$ in the same way into $n-1$ part-tetrahedra.

386*f*. Partition method III. Cut the given tetrahedron into part-tetrahedra by I, and these each into part-tetrahedra by II.

386*g*. Partition method IV. (*Partition with help of Central Projection.*)

Take a point O either *without* the tetrahedron $ABCD$ or coincident with one of its summits, for example A, and let the rays OA, OB, OC, OD meet a plane α in A', B', C', D' which cannot be costraight since A, B, C, D are not coplanar. Calling A' the central projection of A, then the figure made by the central projections of the edges of the tetrahedron $ABCD$ is in general (the special cases are hereafter exhaustively treated) a quadrilateral with its two diagonals, which cut it into triangles. Cut now these last triangles in any way into part-triangles, altogether n in number. The vertices of these part-triangles (in our figures always only one such triangle is shown) join with the projection-centre O, whereby n tetrahedra OT_1', OT_2', . . . OT_n' are made, which have a common summit O, and whose bases T_1', T_2', . . . T_n' are all the part-triangles of the projection-figure $A'B'C'D'$. If O coincides with A then the edges of the tetrahedra OT_1', OT_2', . . . OT_n' cut the original tetrahedron $ABCD$ into n part-tetrahedra.

On the other hand, if O is different from A, then the edges of OT_1', OT_2', ... OT_n' cut the tetrahedron $ABCD$ into a number of *truncated* tetrahedra, among which as special cases may appear the pyramid of five summits and the complete tetrahedron. These pyramids appear if a vertex of the corresponding triangle T' falls on one of the sects $A'B'$. $A'C'$ ... Complete tetrahedra appear if two vertices of the triangle T' fall on one of the said sects.

Always cut these truncated tetrahedra into three (these pyramids into two) complete tetrahedra, using a diagonal of each quadrilateral face [as in 385 II].

By this last partition we get no new summits.

The partition of the original tetrahedron into part-tetrahedra so attained is called *Partition with the aid of central projection.* The different cases may be more explicitly set forth as follows:

Case 1. *O coincides with A.* This is Partition method I [see 386*c*].

Case 2. *O lies on the prolongation of an edge*, for example DA (Fig. 151).

The projection-points A' and D' coincide and we get on the plane α as projection of $ABCD$ a triangle $A'B'C'$.

That face of the truncated tetrahedron (or pyramid of five summits or complete tetrahedron) obtained by the above given construction, which by the projection of T_m' is made in the face BCD designate by T_m; that in face ABC by T_{Im}.

The tetrahedra obtained by the transversal

partition of this truncated tetrahedron designate respectively by t_m', t_m'', t_m'''. Then is the volume of $ABCD$ equal to the sum of the volumes of all the t.

For every tetrahedron OT_m is divided into at most four part-tetrahedra OT_{Im}, t_m', t_m'', t_m''', all

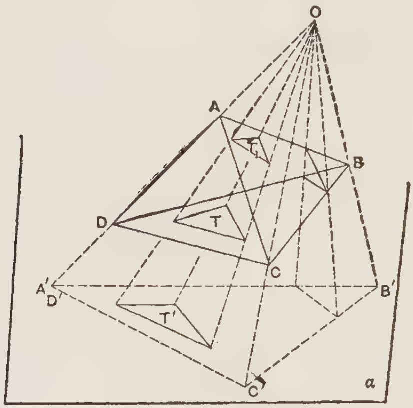

FIG. 151.

of whose summits lie on the three edges meeting in apex O, that is every OT_m is divided by Partition method II. Therefore

$$V(OT_1) = V(OT_{I1}) + V(t_1') + V(t_1'') + V(t_1'''),$$
$$V(OT_2) = V(OT_{I2}) + V(t_2') + V(t_2'') + V(t_2'''),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$V(OT_n) = V(OT_{In}) + V(t_n') + V(t_n'') + V(t_n''').$$

Adding these equations to one another and noticing that (by 386c) on the one side

$$V(OT_1) + V(OT_2) + \ldots + V(OT_n) = V(OBCD),$$

on the other side

$$V(OT_{I1}) + V(OT_{I2}) + \ldots + V(OT_{In}) = V(OABC),$$

we find, using $\sum\limits_{n=1}^{n=n}$ to mean the sum from $n=1$ up to $n=n$,

$$V(OBCD)=V(OABC)+\sum_{n=1}^{n=n}[V(t_n')+V(t_n'')+V(t_n''')].$$

Furthermore, the division of the tetrahedron $OBCD$ by the plane ABC into the tetrahedra $OABC$ and $ABCD$ is a transversal partition, and so (by 384)

$$V(OBCD)=V(OABC)+V(ABCD).$$

The last two equations give finally

$$V(ABCD)=\sum_{n=1}^{n=n}[V(t_n')+V(t_n'')+V(t_n''')];$$

therefore also for this case our theorem is proven.

Case 3. *O lies without the tetrahedron on one of its boundary planes*, that is in the plane of one of its faces; for example, on the plane ABD (Fig. 152).

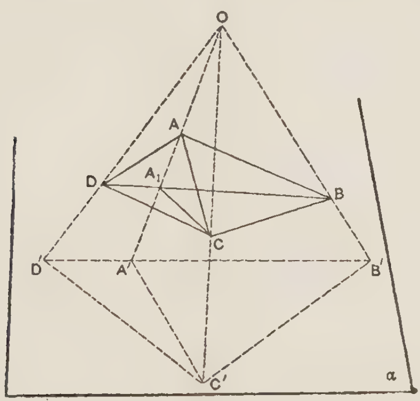

FIG. 152.

Then the projections A', B', D' are costraight; therefore one of them, say A', lies between the other two.

B', C', D' make a triangle which is cut by the sect $A'C'$ into two part-triangles $A'B'C'$ and $A'C'D'$. Correspondingly the plane $OA'C'$ cuts the tetrahedron $ABCD$ into two part-tetrahedra AA_1BC, and AA_1CD; and since this plane goes through the edge AC, therefore it makes a transversal partition, and we have (by 384)

$$V(ABCD) = V(AA_1BC) + V(AA_1CD).$$

Since, however, O lies on the prolongation of the edge A_1A, therefore (by Case 2)

$$V(AA_1BC) = \Sigma V(t_b)$$

and

$$V(AA_1CD) = \Sigma V(t_d),$$

where t_b are all the tetrahedra into which in accordance with the above method the tetrahedron AA_1BC is divided [and t_d those in AA_1CD].

These last three equations give now $V(ABCD) = \Sigma V(t)$, where t are all the part-tetrahedra of $ABCD$.

Case 4. If *O lies on no one of the boundary planes of the tetrahedron $ABCD$*, then no three of the projection-points A', B', C', D' are costraight.

Consider first the case in which *one of these points (say A') falls within the triangle $B'C'D'$ made by the other three* (Fig. 153).

The joining sects $A'B'$, $A'C'$, $A'D'$ cut the triangle $B'C'D'$ into three part-triangles, and correspondingly the tetrahedron $ABCD$ is divided into three part-tetrahedra AA_1BC, AA_1BD, AA_1CD,

giving a case of Partition method I; therefore (by 386*c*)

$$V(ABCD) = V(AA_1BC) + V(AA_1BD) + V(AA_1CD).$$

If we now, as above, divide the triangles $A'B'C'$, $A'B'D'$, $A'C'D'$ into part-triangles, project these

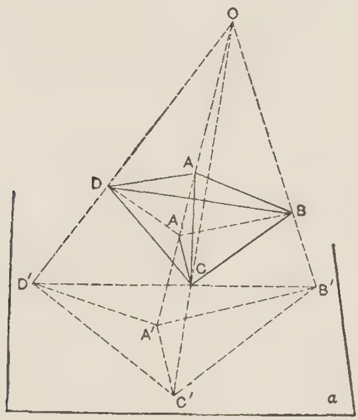

FIG. 153.

back upon the corresponding part-tetrahedra, and designate the part-tetrahedra obtained in the above given way of the tetrahedra AA_1BC, AA_1BD, AA_1CD with t_d, t_c, t_b respectively, then we obtain the three equations

$$V(AA_1BC) = \Sigma V(t_d); \ V(AA_1BD) = \Sigma V(t_c);$$
$$V(AA_1CD) = \Sigma V(t_b),$$

since O lies on the ray A_1A, that is we have here each time Case 2.

These last four equations give now $V(ABCD) = \Sigma V(t)$, where t are all the part-tetrahedra of $ABCD$.

Case 5. If, finally, *again O lies in no one of the boundary planes of the tetrahedron $ABCD$, but each of the projections A', B', C', D' falls without the triangle made by the remaining projection-points,* then the projections of the edges of the tetrahedron make a convex quadrilateral with its two diagonals. This quadrilateral is divided by its diagonals into four triangles, $M'A'C'$, $M'A'D'$, $M'B'C'$, $M'B'D'$ (Fig. 154).

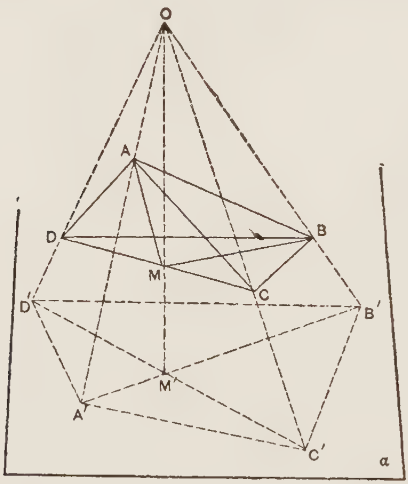

FIG. 154.

The plane $OA'B'$ divides the tetrahedron by a transversal partition into two parts, so that we have (by 384)

$$V(ABCD) = V(AMBC) + V(AMBD),$$

where M is the point corresponding to the intersection point M' of the diagonals $A'B'$ and $C'D'$.

If we now divide each triangle $M'A'C'$, $M'A'D'$, $M'B'C'$, $M'B'D'$ into part-triangles and project back upon the faces of the tetrahedron $ABCD$, then the tetrahedra $AMBC$ and $AMBD$ are divided each into part-tetrahedra, which, in general, may be designated respectively t_c and t_d. Since now, moreover, O lies in the plane AMB, therefore (by Case 3) we have the equations

$$V(AMBC) = \Sigma V(t_c) \quad \text{and} \quad V(AMBD) = \Sigma V(t_d).$$

Consequently is also in this case

$$V(ABCD) = \Sigma V(t).$$

This completes the proof of the theorem that in every partition of a tetrahedron by central projection the sum of the volumes of the part-tetrahedra equals the volume of the whole tetrahedron.

386*h*. *The most general partition can be built up from the four partition methods already given*, and this proves the fundamental theorem 386.

For let $ABCD$ be a tetrahedron and P_1, P_2, . . . P_k the part-tetrahedra which arise from any partition of it.

If, now, we project all these tetrahedra from the point A upon the face BCD, then their projections, which necessarily all fall within the triangle BCD, overlie and overlap, in general, manifoldly, and cut one another into polygons. When we cut these polygons into triangles and join their vertices with A we divide each tetrahedron, $P_m (m = 1, 2, \ldots k)$, into a number of truncated tetrahedra (including perhaps pyramids of five summits and

complete tetrahedra) which in turn in the well-known way we divide into further part-tetrahedra.

Since the so obtained partition of each part-tetrahedron P_m into further part-tetrahedra, which may be designated t_m', t_m'', . . . $t_m^{(n_m)}$, is accomplished with the aid of central projection, for each projection-center lies without each tetrahedron, only that one with the summit A excepted, so is

$$V(P_m) = V(t_m') + V(t_m'') + \ldots + V(t_m^{(n_m)}).$$

If we now give m successively the values 1, 2, . . . k, we obtain k such equations, which added give the following:

$$\Sigma V(P_m) = \Sigma V(t_m^{(l)}) \left(\begin{matrix} m = 1, 2, \ldots k \\ l = 1, 2, \ldots n_m \end{matrix} \right).$$

But, on the other hand, every tetrahedron AT, where T is a part-triangle of BCD, cuts out from the aggregate of part-tetrahedra t a set, and each tetrahedron of this set appears once and only once in the above sum.

At the same time all summits of these last part-tetrahedra lie on the three edges from A of the particular tetrahedron AT; that is, AT is divided by this set of tetrahedra according to Partition method II.

Furthermore, the whole tetrahedron $ABCD$ is divided into tetrahedra AT according to Partition method I, so that it is divided according to Partition method III into part-tetrahedra t.

Now this complex of tetrahedra t is identical

with the complex $t_m^{(l)}$, where $l = 1, 2, \dots n_m$; $m = 1, 2, \dots k$.

Consequently

$$V(ABCD) \doteq \Sigma V(t_m^{(l)}) \binom{m=1,2,\dots k}{l=1,2\dots n_m},$$

which combined with the previous equation gives the desired proof of the fundamental theorem

$$V(ABCD) = \sum_{m=1}^{m=k} V(P_m).$$

387. Theorem. *If a polyhedron P is cut into tetrahedra in two different ways, then the sum of the volumes of the tetrahedra of the first partition equals that of the second.*

Proof. Suppose P divided into m tetrahedra $t_1, t_2, \dots t_m$, and again into n tetrahedra $t_1', t_2', \dots t_n'$.

Construct a tetrahedron T which contains the polyhedron P, and cut the polyhedron bounded by the surface of P, and that of T in any definite way into tetrahedra $T_1', T_2', \dots$

Thus we obtain two partitions of the tetrahedron T and (by 386) the equations

$$V(T) = V(t_1) + V(t_2) + \dots + V(t_m) + V(T_1') + V(T_2') + \dots$$

$$V(T) = V(t_1') + V(t_2') + \dots + V(t_n') + V(T_1') + V(T_2') + \dots$$

whence

$$V(t_1) + V(t_2) + \dots + V(t_m) = V(t_1') + V(t_2') + \dots + V(t_n').$$

388. Definition. The volume of a polyhedron is the sum of the volumes of any set of tetrahedra into which it is cut.

389. Definition. Two polyhedra P and Q of equal volume are said to be equal. P is called greater than Q if the volume of P is greater than the volume of Q; less if volume is less.

If $V(P) = V(Q)$, we say $P = Q$.

If $V(P) > V(Q)$, we say $P > Q$.

If $V(P) < V(Q)$, we say $P < Q$.

The three cases are mutually exclusive.

390. Corollary. If a polyhedron be cut into polyhedra, the sum of their volumes equals its volume.

391. Corollary. If a polyhedron be cut into polyhedra, if one of these be omitted it is not possible with the others, however arranged, to make up the original polyhedron.

The Prismatoid Formula.

392. Definition. A *prismatoid* is a polyhedron having for base and top any two polygons in parallel planes, and whose lateral faces are triangles determined by the vertices so that each lateral edge with the succeeding forms a triangle with one side of the base or of the top.

The altitude of a prismatoid is the perpendicular from top to base.

A number of different prismatoids thus have the same base, top, and altitude.

A prismatoid with a point as top is a pyramid. If both base and top of a prismatoid are sects, it is a tetrahedron.

If a side of the base and a side of the top which form with the same lateral edge two sides of two adjoining faces are parallel, then these two triangular faces fall in the same plane, and together form a trapezoid.

393. A *prismoid* is a prismatoid whose base and top have the same number of sides, and every corresponding pair parallel.

394. A *frustum of a pyramid* is a prismoid with base and top similar.

395. Corollary. Every prismoid with triangular base is the frustum of a pyramid.

396. A *section* of a prismatoid is the polygon determined by a plane perpendicular to the altitude.

397. Theorem. *The area of a section of a pyramid is to the area of the base as the square of the perpendicular on it, from the apex, is to the square of the altitude of the pyramid.*

To prove $S:B=p^2:a^2$.

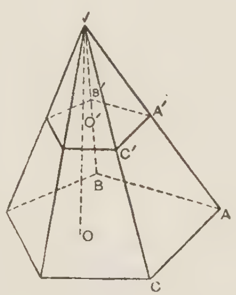

FIG. 155.

Proof. The section and base are similar, since corresponding diagonals cut them into triangles similar in pairs because having all their sides respectively proportional, each corresponding pair being as a lateral edge to the sect on it made by apex and section, which in turn are as altitude to perpendicular on section. But (by 300) the areas are as the squares of these.

398. To find the volume of any pyramid.

Rule. *Multiply one-third its altitude by the area of its base.* Formula, $Y = \frac{a}{3}B$.

Proof. Cutting this base into triangles by planes through the apex, we have a partition of the given pyramid into triangular pyramids (tetrahedra) of the same altitude whose bases together make the polygonal base.

399. To find the volume of any prismatoid.

Rule. *Multiply one-fourth its altitude by the sum of the base and three times a section at two-thirds the altitude from the base.*

Formula, $D = \frac{a}{4}(B + 3S)$.

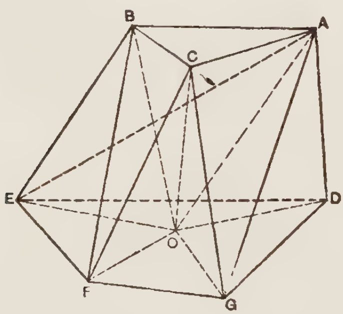

FIG. 156.

Proof. Any prismatoid may be divided into tetrahedra, all of the same altitude as the prismatoid; some, as $CFGO$, having their apex in the top of the prismatoid and their base within its base; some, as $OABC$, having three summits within the top and the fourth in the base of the

prismatoid, thus having for base a point and for top a triangle; and the others, as $ACOG$, having for base and top a pair of opposite edges, a sect in the plane of the base and a sect in the plane of the top of the prismatoid, as OG and AC.

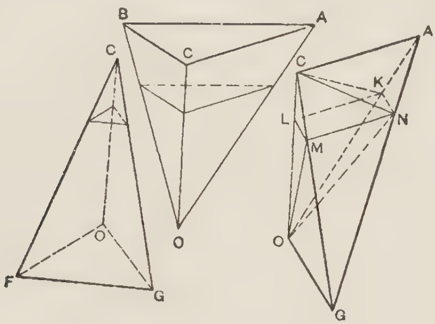

Fig. 157.

Therefore if the formula holds good for tetrahedra in these three positions, it holds for the prismatoid, their sum.

In (1) call S the section at two-thirds the altitude from the base B_1; then S_1 is $\frac{1}{3}a$ from the apex. Therefore (by 397) the areas

$$S_1 : B_1 = (\tfrac{1}{3}a)^2 : a^2, \ \therefore S_1 = \tfrac{1}{9}B_1;$$

$$\therefore D_1 = \frac{a}{4}(B_1 + 3S_1) = \frac{a}{4}(B_1 + \tfrac{1}{3}B_1) = \tfrac{1}{3}aB_1,$$

which (by 382) equals Y_1, the volume of this tetrahedron.

In (2) the base B_2 is a point, and S_2 is $\frac{2}{3}a$ from

this point, which is the apex of an inverted pyramid.

$\therefore$ (by 397) the areas $S_2 : T_2 = (\frac{2}{3}a)^2 : a^2$; $\therefore S_2 = \frac{4}{9}T_2$;

$$\therefore D_2 = \frac{a}{4}(B_2 + 3S_2) = \frac{a}{4}(0 + \tfrac{4}{3}T_2) = \tfrac{1}{3}aT_2 = Y_2.$$

In (3) let $KLMN$ be the section S_3. Now the areas

$$\triangle ANK : \triangle AGO = \overline{AN}^2 : \overline{AG}^2 = (\tfrac{1}{3}a)^2 : a^2 = 1:9;$$

$$\triangle GNM : \triangle GAC = \overline{GN}^2 : \overline{GA}^2 = (\tfrac{2}{3}a)^2 : a^2 = 4:9.$$

But the whole tetrahedron D_3 and the pyramid $CANK$ may be considered as having their bases in the same plane, AGO, and the same altitude, a perpendicular from C:

$$\therefore CANK : D_3 = \triangle ANK : \triangle AGO = 1:9;$$

$$\therefore CANK = \tfrac{1}{9}D_3.$$

In the same way

$$OGNM : D_3 = \triangle GNM : \triangle GAC = 4:9;$$

$$\therefore OGNM = \tfrac{4}{9}D_3;$$

$$\therefore CANK + OGNM = \tfrac{5}{9}D_3;$$

$$\therefore CKLMN + OKLMN = \tfrac{4}{9}D_3.$$

But by (398) $CKLMN + OKLMN = \frac{1}{3}\cdot\frac{1}{3}aS_3 + \frac{1}{3}\cdot\frac{2}{3}aS_3 = \frac{1}{3}aS_3$; $\therefore \frac{4}{9}D_3 = \frac{1}{3}aS_3$;

$$\therefore D_3 = \frac{a}{4}3S_3 = \frac{a}{4}(B_3 + 3S_3),$$

since here the area $B_3 = 0$.

400. Corollary to 399. Since, in a frustum of a pyramid, B and S are similar; $\therefore$ if b and s be corresponding edges,

$$B:S=b^2:s^2; \quad \therefore \text{ the volume } F=\frac{a}{4}B\left(1+\frac{3s^2}{b^2}\right).$$

401. Definition. A *prism* is a prismatoid whose base and top are congruent.

A *right prism* is one whose lateral edges are perpendicular to its base.

A *parallelopiped* is a prism whose bases are prallelograms.

A *cuboid* is a parallclopiped whose six faces are rectangles.

A *cube* is a cuboid whose six faces are squares.

402. Corollary to 400. To find the volume of any prism.

Rulc. *Multiply its altitude by the area of its base.* Formula, $V(P)=a\cdot B$.

$\therefore$ To find the volume of a cuboid.

Rule. *Multiply together any three copunctal edges*, that is, its length, breadth, and thickness.

403. A cube whose edge is the unit sect has for volume this unit sect, since $1\times 1\times 1=1$.

Any polyhedron has for volume as many such unit sects as the polyhedron contains such cubes on the unit sect.

The number expressing the volume of a polyhedron will thus be the same in terms of our unit sect, or in terms of a cube on this sect, considered as a new kind of unit, a unit solid. Such units, though traditional, are unnecessary.

Ex. 525. If the altitude of the highest Egyptian pyramid is 138 meters, and a side of its square base 228 meters, find its volume.

Ex. 526. The pyramid of Memphis has an altitude of 73 Toises; the base is a square whose side is 116 Toises. If a Toise is 1·95 meters, find the volume of this pyramid.

Ex. 527. A pyramid of volume 15 has an altitude of 9 units. Find the area of its base.

Ex. 528. Find the volume of a rectangular prismoid of 12 meters altitude, whose top is 5 meters long and 2 meters broad, and base 7 meters long and 4 meters broad.

Ex. 529. In a prismoid 15 meters tall, whose base is 36 square meters, each basal edge is to the top edge as 3 to 2. Find the volume.

Ex. 530. Every regular octahedron is a prismatoid whose bases and lateral faces are all congruent equilateral triangles. Find its volume in terms of an edge b.

Ex. 531. The bases of a prismatoid are congruent squares of side b, whose sides are not parallel; the lateral faces are eight isosceles triangles. Find the volume.

Ans. $\frac{1}{3}ab^2(2+2^{\frac{1}{2}})$.

Ex. 532. If from a regular icosahedron we take off two five-sided pyramids whose vertices are opposite summits, there remains a solid bounded by two congruent regular pentagons and ten equilateral triangles. Find its volume from an edge b. *Ans.* $\frac{1}{6}b^3[5+2(5)^{\frac{1}{2}}]$.

Ex. 533. Both bases of a prismatoid of altitude a are squares; the lateral faces isosceles triangles. The sides of the upper base are parallel to the diagonals of the lower base, and half as long as these diagonals; and b is a side of the lower base. Find the volume. *Ans.* $\frac{5}{6}ab^2$.

Ex. 534. The upper base of a prismatoid of altitude $a=6$ is a square of side, $b_2=7·07107$; the lower base is a square of side $b_1=10$, with its diagonals parallel to sides of the upper base; the lateral faces are isosceles triangles. Find volume.

Ex. 535. Every prismatoid is equal in volume to three

pyramids of the same altitude with it, of which one has for base half the sum of the prismatoid's bases, and each of the others its mid-cross section:

$$D = \tfrac{1}{3}a\left(\frac{B_1 + B_2}{2} + 2M\right) = \tfrac{1}{6}a(B_1 + 4M + B_2).$$

Ex. 536. If a prismatoid have bases with angles respectively equal and their sides parallel, in volume it equals a prism plus a pyramid, both of the same altitude with it, whose bases have the same angles as the bases of the prismatoid, but the basal edges of the prism are half the sum, and of the pyramid half the difference, of the corresponding sides of both the prismatoid's bases.

Ex. 537. If the bases of a prismatoid are trapezoids whose mid-sects are b_1 and b_2, and whose altitudes are a_1 and a_2, the volume $= a\left(\frac{a_1 + a_2}{2} \cdot \frac{b_1 + b_2}{2} + \frac{1}{3} \frac{a_1 - a_2}{2} \cdot \frac{b_1 - b_2}{2}\right)$.

Ex. 538. A side of the base of a frustum of a square pyramid is 25 meters, a side of the top is 9 meters, and the height is 240 meters. Required the volume of the frustum.

Ex. 539. The sides of the square bases of a frustum are 50 and 40 centimeters. Each lateral edge is 30 centimeters. How many liters would it contain?

Ex. 540. In the frustum of a pyramid whose base is 50 and altitude 6, the basal edge is to the corresponding top edge as 5 to 3. Find volume.

Ex. 541. Near Memphis stands a frustum whose height is 142·85 meters, and bases are squares on edges of 185·5 and 3·714 meters. Find its volume.

Ex. 542. In the frustum of a regular tetrahedron, given a basal edge, a top edge, and the volume. Find the altitude.

Ex. 543. A wedge of 10 centimeters altitude, 4 centimeters edge, has a square base of 36 centimeters perimeter. Find volume.

Ex. 544. The diagonal of a cube is n. Find its volume.

Ex. 545. The edge of a cube is n. Approximate to the edge of a cube twice as large.

Ex. 546. Find the cube whose volume equals its superficial area.

Ex. 547. Find the edge of a cube equal to three whose edges are a, b, l.

Ex. 548. If a cubical block of marble, of which the edge is 1 meter, costs one dollar, what costs a cubical block whose edge is equal to the diagonal of the first block?

Ex. 549. If the altitude, breadth, and length of a cuboid be a, b, l, and its volume V,

(1) Given a, b, and superficial area; find V.

(2) Given a, b, V; find l.

(3) Given V, (ab), (bl); find l and b.

(4) Given V, $\left(\frac{a}{b}\right)$, $\left(\frac{b}{l}\right)$; find a and b.

(5) Given (ab), (al), (bl); find a and b.

Ex. 550. If 97 centimeters is the diagonal of a cuboid with square base of 43 centimeters side, find its volume.

Ex. 551. The volume of a cuboid whose basal edges are 12 and 4 meters is equal to the superficial area. Find its altitude.

Ex. 552. In a cuboid of 360 superficial area, the base is a square of edge 6. Find the volume.

Ex. 553. A cuboid of volume 864 has a square base equal in area to the area of two adjacent sides. Find its three dimensions.

Ex. 554. In a cuboid of altitude 8 and superficial area 160 the base is square. Find the volume.

Ex. 555. The volume of a cuboid is 144, its diagonal 13, the diagonal of its base 5. Find its three dimensions.

Ex. 556. In a cuboid of surface 108, the base, a square, equals in area the area of the four sides. Find volume.

Ex. 557. What is the area of the sheet of metal required to construct a rectangular tank (open at top) 12 meters long, 10 meters broad, and 8 meters deep?

Ex. 558. The base of a prism 10 meters tall is an isosceles right triangle of 6 meters hypothenuse. Find volume.

Ex. 559. In a prism the area of whose base is 210 the three sides are rectangles of area 336, 300, 204. Find volume.

Ex. 560. A right prism whose volume is 480 stands upon an isosceles triangle whose base is 10 and side 13. Find altitude.

Ex. 561. In a right prism whose volume is 54, the lateral area is four times the area of the base, an equilateral triangle. Find basal edge.

Ex. 562. The vertical ends of a hollow trough are parallel equilateral triangles with 1 meter in each side, a pair of sides being horizontal. If the length between the triangular ends be 6 meters, find the volume of water the trough will contain.

CHAPTER XIII.

TRIDIMENSIONAL SPHERICS.

404. Definition. If C is any given point, then the aggregate of all points A for which the sects CA are congruent to one another is called a *sphere*. C is called the center of the sphere, and CA the radius.

Every point B, such that $CA > CB$ is said to be *within* the sphere. If $CA < CD$, then D is *without* the sphere.

405. Theorem. Any ray from the center of a sphere cuts the sphere in one, and only one, point.

406. Theorem. Any straight through its center cuts the sphere in two, and only two, points.

407. Definition. A sect whose end-points are on the sphere is called a chord.

408. Definition. Any chord through the center is called a diameter. Its end-points are called *opposite points* of the sphere.

409. Theorem. Every diameter is bisected by the center.

410. Corollary to 106 and 409. A plane through its center meets the sphere in a circle with radius equal to that of the sphere. Such a circle is called a *great circle* of the sphere.

411. Corollary to 410. All great circles of the sphere are congruent, since each has for its radius the radius of the sphere.

412. Theorem. *Any two great circles of a sphere bisect each other.*

Proof. Since the planes of these circles both pass through the center of the sphere, therefore on their intersection is a diameter of the sphere which is a diameter of each circle.

413. Theorem. *If any number of great circles pass through a given point, they will also pass through the opposite point.*

Proof. The given point and the center of the sphere determine the same diameter for each of the circles.

414. Corollary to 413. Through opposite points an indefinite number of great circles can be passed.

415. Theorem. *Through any two non-opposite points on a sphere, one, and only one, great circle can be passed.*

Proof. For the two given points and the center of the sphere determine its plane.

416. Definition. A straight or plane is called tangent to a sphere when it has one point, and only one, in common with the sphere.

Two spheres are called tangent to each other when they have one point, and only one, in common.

417. Theorem. *A straight or plane having the foot of the perpendicular to it from the center in common with the sphere is tangent.*

Proof. This perpendicular, a radius, is (by 142)

less than any other sect from the center to this straight or plane. Therefore every point of the straight or plane is without the sphere except the foot of this radius.

418. Theorem. *If a straight has a given point not the foot of the perpendicular to it from the center in common with a sphere, it has a second point on the sphere.*

Proof. This is the other end-point of the sect from the given point bisected by this perpendicular.

419. Theorem. *If a plane has a point not the foot of the perpendicular to it from the center in common with a sphere, it cuts the sphere in a circle.*

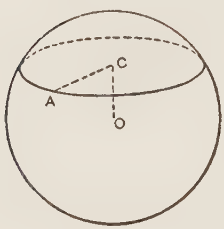

Fig. 158.

Proof. If A be the common point and C the foot of the perpendicular, the circle $\odot C(CA)$ is on the sphere.

420. Corollary to 419. The straight through the center of any circle of a sphere perpendicular to its plane passes through the center of the sphere.

421. Definition. The two opposite points in which the perpendicular to its plane, through the center of a circle of the sphere, meets the sphere, are called the *poles* of that circle, and the diameter between them its *axis*

422. Theorem. *Any three points on a sphere determine a circle on the sphere* (I 3 and 419).

423. Theorem. The radius of any circle of the sphere whose plane does not contain the center

of the sphere is less than the radius of a great circle.

Proof. The hypothenuse (by 142) is > a side.

424. Definition. A circle on the sphere whose plane does not contain the center of the sphere is called a *small circle* of the sphere.

425. Inverse of 417. Every straight or plane tangent to the sphere is perpendicular to the radius at the point of contact. For if not it would have (by 418) another point on it.

426. Theorem. *If two spheres have two points in common they cut in a circle whose center is in their center-straight and whose plane is perpendicular to that straight.*

Hypothesis. Let C and O be the centers of the spheres having the points A and B in common.

Conclusion. They have in common all points, and only those, on a circle with center on OC and plane $\perp$ to OC.

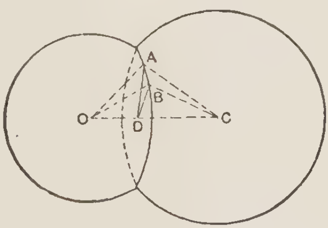

FIG. 159.

Proof. Since, by 58, $\triangle ACO \equiv \triangle BCO$, $\therefore$ perpendiculars from A and B upon OC are equal and meet OC at the same point, D. Thus all, but only, points like A and B, in a plane $\perp$ to OC, and points of $\odot D(DA)$, are on both spheres.

427. Corollary to 426. If two spheres are tangent, either internally or externally, their centers and point of contact are costraight.

428. Theorem. *Four points, not coplanar, determine a sphere.*

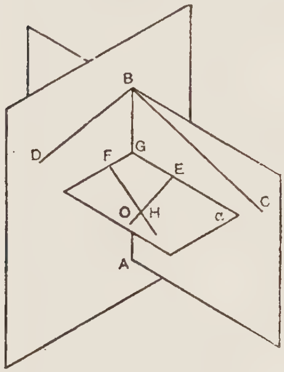

Fig. 160.

Proof. Let A, B, C, D be the four given points. Then (by 352) the plane $\alpha \perp$ to and bisecting AB meets $\beta \perp$ to and bisecting BC in $EH \perp ABC$, and meets $\delta \perp$ to and bisecting BD in $FO \perp ABD$.

$\therefore EH \perp EG$, and $FO \perp FG$, and (by 77) EH and FO meet, say, at O; $\therefore$ (by 346) O is one, and the only center of a sphere containing A, B, C, D.

429. Corollary to 428. The four perpendiculars to the faces of a tetrahedron through their circumcenters, and the six planes bisecting at right angles the edges, are copunctal in its circumcenter.

430. Problem. *To inscribe a sphere in a given tetrahedron.*

Construction. Through any edge and any point from which perpendiculars to its two faces are equal, take a plane. Likewise with the other edges in the same face. The cointersection of these three planes is the incenter required.

431. Theorem. *The sects joining its pole to points on any circle of the sphere are equal.*

Proof (343).

432. Corollary to 431. Since equal chords have congruent minor arcs, ∴ the great-circle-arcs joining a pole to points on its circle are congruent. Hence if C is any point in a sphere α, then the aggregate of all points A in α, for which the great-circle-arcs CA are congruent to one another is a circle.

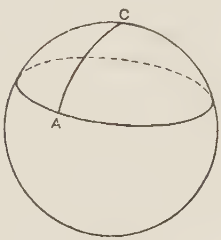

Fig. 161.

433. Theorem. *The great-circle-arc joining any point in a great circle with its pole is a quadrant.*

Proof The angle at the center is right.

434. Theorem. *If A, B are non-opposite, the point P is a pole of their great circle when the arcs PA, PB are both great-circle-quadrants.*

For each of the angles POA, POB is right and ∴ $PO \perp OAB$.

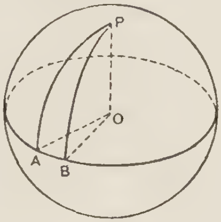

Fig. 162.

435. Definition. The angle between two great-circle-arcs on a sphere, called a *spherical angle*, is the angle between tangents to those arcs at their point of meeting.

436. A spherical angle is the inclination of the two hemiplanes containing the arcs.

437. Theorem. *Any great circle through a pole of a given great circle is perpendicular to the given great circle.*

Proof. Their planes (by 369) are at right angles.

438. Inverse of 437. Any great circle perpen-

dicular to a given great circle will pass through its poles.

439. Theorem. *If a sphere be tangent to the parallel planes containing opposite edges of a tetrahedron, and sections made in the sphere and tetrahedron by one plane parallel to these are of equal area, so are sections made by any parallel plane.*

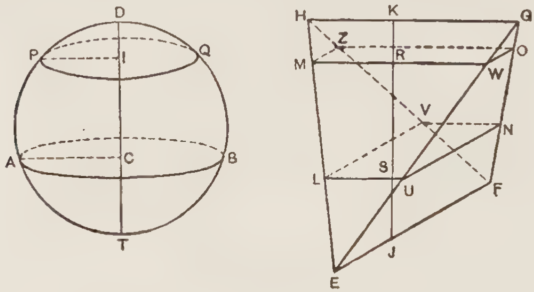

FIG. 163.

Hypothesis. Let KJ be the sect $\perp$ to the edges EF and GH in the $\parallel$ tangent planes. Then $KJ = DT$, the diameter.

Let $\odot I(IP) = MO$, sections made by the plane $\perp DT$ at I and $\perp KJ$ at R, where $KR = DI$.

Let $ABCLSN$ be any parallel plane through a point A of the sphere.

Conculsion. $LN = \odot C(CA)$.

Proof. Since

$$\triangle LEU \sim \triangle MEW, \quad \text{and} \quad \triangle LHV \sim \triangle MHZ,$$

$$\therefore MW : LU = EM : EL = JR : JS \text{ (by 373).}$$

$$MZ : LV = HM : HL = KR : KS.$$

But (by 366) $\measuredangle ZMW = \measuredangle VLU$.

$\therefore$ (by 299) area MO : area $LN = WM : MZ : UL \cdot LV$

$$= JR \cdot RK : JS \cdot SK.$$

But (by 325)

$$\text{area } \odot I(IP):\text{area } \odot C(CA) = \overline{PI}^2:AC^2$$
$$= TI\cdot ID:TC\cdot CD \text{ (by 245)};$$

$\therefore$ area LN = area $\odot C(CA)$.

440. Cavalieri's assumption. If the two sections made in two solids between two parallel planes by any parallel plane are of equal area, then the solids are of equal volume.

441. Theorem. *The volume of a sphere of radius* r *is* $\frac{4}{3}\pi r^3$.

Proof. A tetrahedron on edge, and a sphere with this tetrahedron's altitude for diameter, have (by 439) all their corresponding sections of equal area, if any one pair are of equal area.

Hence (by 440) they are of equal volume.

$\therefore$ (by 399) vol. sphere $= \frac{3}{4}aS$.

But $a = 2r$, and (by 245) $S = \frac{2}{3}r\cdot\frac{4}{3}r\cdot\pi$.

$\therefore$ Vol. sphere $= \frac{3}{4}\cdot 2r\cdot\frac{2}{3}r\cdot\frac{4}{3}r\cdot\pi = \frac{4}{3}\pi r^3$.

442. Definition. The area of a sphere is the quotient of its volume by one-third its radius.

Area of sphere $= 4\pi r^2$.

443. Corollary to 324. The area of a sphere is quadruple the area of its great circle.

444. Definition. A *spherical segment* is the piece of a sphere between two parallel planes. If one of the parallel planes is tangent to the sphere, the segment is called a *segment of one base.*

445. Corollary to 439 and 440. The volume of a spherical segment is $\frac{a}{4}\pi(r_1{}^2 + 3r_3{}^2)$, where r_3 is the

radius of the section two-thirds the altitude from the base whose radius is r_1. If the segment is of one base its volume is $\frac{3}{4}a\pi r_3{}^2$; which in terms of r, the radius of the sphere, is $\pi a^2\left(r-\frac{a}{3}\right)$, and equals $\frac{1}{2}\pi a\left(r_2{}^2+\frac{a^2}{3}\right)$. If we eliminate r_3 by introducing r_2, the radius of the top, the volume of the segment is $\frac{1}{6}\pi a[3(r_1{}^2+r_2{}^2)+a^2]$.

446. Problem. Given a portion of a sphere, *find its radius.*

Construction. Take any three points of the part given, say A, B, C. The plane A, B, C (by 419) cuts the sphere in a circle. The straight at D, the center of this circle perpendicular to the plane ABC, contains the center O of the sphere (by 420) and therefore meets the sphere, say at P. In the plane PAD draw $AP' \perp$ to AP and meeting DO in P'. Bisect PP' in O. Then O is the center of the sphere and OP is the radius.

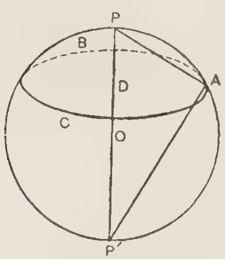

FIG. 164.

Proof: O is circumcenter of PAP'.

$\therefore OP=OA$. But since OD is $\perp \odot (D) DA$ at D, $\therefore OA=OB=OC$.

447. Corollary to 305.

$$OP=\frac{\overline{DA}^2+\overline{DP}^2}{2DP}, \text{ that is, } R=\frac{r^2+h^2}{2h}.$$

Ex. 563. A circle on a sphere of 10 centimeters radius

has its center 8 centimeters from the center of the sphere. Find its radius.

Ex. 564. The sects from the centers of circles of equal area on a sphere to the center of the sphere are equal.

Ex. 565. Where are the centers of spheres through three given points?

Ex. 566. Find the volume of a sphere whose area is 20.

Ex. 567. Find the radius of a globe equal to the sum of two globes whose radii are 3 and 6 centimeters.

Ex. 568. A section parallel to the base of a hemisphere, radius 1, bisects its altitude. Find the volume of each part.

Ex. 569. The areas of the parts into which a sphere is cut by a plane are as 5 to 7. To what numbers are the volumes of these parts proportional?

Ex. 570. The volume of a spherical segment of one base and height 8 is 1200. Find radius of the sphere.

Ex. 571. Find the volume of a segment of 12 centimeters altitude, the radius of whose single base is 24 centimeters.

Ex. 572. In terms of sphere radius, find the altitude of a spherical segment n times its base.

Ex. 573. Find volume of a spherical segment of one base whose area is 15 and base 2 from sphere center.

Ex. 574. In a sphere of 10 centimeters radius find the radii r_1 and r_2 of the base and top of a segment whose altitude is 6 centimeters and base 2 centimeters from the sphere center.

Ex. 575. Out of a sphere of 12 centimeters radius is cut a segment whose volume is one-third that of the sphere and whose bases are congruent. Find the radius of the bases.

Ex. 576. Find the radius of a sphere whose area equals the length of a great circle.

Ex. 577. Find the volume of a sphere the length of whose great circle is n.

Ex. 578. Find the radius of a sphere whose volume equals the length of a great circle.

Ex. 579. The volume of a sphere is to that of the circumscribing cube as π to 6.

Ex. 580. Find altitude of a spherical segment of one base if its area is A and the volume of the sphere V.

Ex. 581. The radii of the bases of a spherical segment are 5 and 4; its altitude 3. Find volume.

CHAPTER XIV.

CONE AND CYLINDER.

448. Definition. The aggregate of straights determined by pairing the points of a circle each with the same point not in their plane is called a *circular cone of two nappes.*

This point is called the *apex* of the cone. Each straight is an *element.*

The straight determined by the apex and the center is called the *axis* of the cone.

The rays of the cone on the same side of a plane through the apex perpendicular to the axis are one nappe of the cone.

The sects from the apex to the circle are often called the cone, and are meant when we speak of the area or the volume of the cone.

When each element makes the same angle with the axis, the cone is called a *right* cone.

In a right cone all sects from apex to circle are equal, and each is called the *slant height.*

449. Theorem. *Every section of a circular cone by a plane parallel to the base is a circle.*

Let the section $D'H'B'F'$ of the circular cone $A\text{–}DHBF$ be parallel to the base.

To prove $D'H'B'F'$ a circle.

Proof. Let C be the center of the base, and

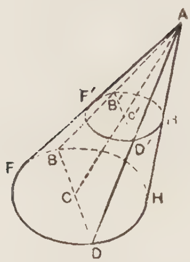

FIG. 165.

C' the point of the axis AC in the plane $D'H'B'$. The plane through AC and any element AB gives radii CB, CD, and parallel to them the sects $C'B'$, $C'D'$.

$\therefore$ (by 75) $\triangle ABC \sim \triangle AB'C'$ and $\triangle ACD \sim \triangle AC'D'$.

$\therefore$ (by 234) $C'B' : CB = AC' : AC = C'D' : CD$.

But $CB = CD$. $\therefore C'B' = C'D'$.

450. Corollary to 449.

The axis of a circular cone passes through the center of every section parallel to the base.

451. Theorem. *If a circular cone and a tetrahedron have equal altitudes and bases of equal area and in the same plane, sections by a plane parallel to the bases are of equal area.*

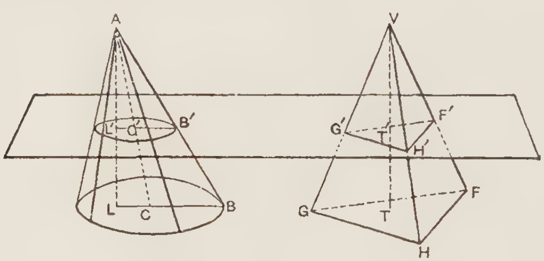

FIG. 166.

Proof. $BC : B'C' = AC : AC' = AL : AL' =$
$= VT : VT' = VH : VH' = GH : G'H'$.

$\therefore \overline{BC}^2 : \overline{B'C'}^2 = \overline{GH}^2 : \overline{G'H'}^2$

But (by 325)

area $\odot C(CB)$: area $\odot C'(C'B')$

$= \overline{BC}^2 : B'C'^2 = \overline{GH}^2 : \overline{G'H'}^2$

$=$ area $\triangle FGH$: area $\triangle F'G'H'$ (by 300).

But by hypothesis area $\odot C(CB) =$ area $\triangle FGH$.

$\therefore$ Area $\odot C'(C'B') =$ area $\triangle F'G'H'$.

452. Corollary to 451.

Volume of circular cone is (by 440) = volume of tetrahedron of equal altitude and base $= \frac{1}{3}a\pi r^2$.

453. Theorem. *The lateral area of a right circular cone is half the product of the slant height by the length of the base.*

Proof. It has the same area as a sector of a circle with the slant height as radius and an arc equal in length to the length of the cone's base.

$\therefore$ (by 323) $K = \frac{1}{2}ch = \pi rh$.

454. Definition. A *truncated* pyramid or cone is the portion included between the base and a plane meeting all the elements.

A *frustum* of a cone is the portion included between the base and a plane parallel to the base.

455. Theorem. *The lateral area of a frustum of a right circular cone is half the product of its slant height by the sum of the lengths of its bases.*

Proof. It is the difference of the areas of two sectors with a common angle, the lengths of the arcs of the sectors being equal to the lengths of bases of the frustum.

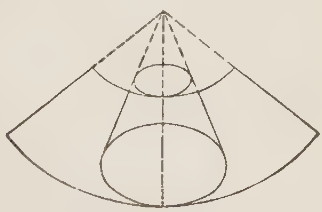

Fig. 167.

$\therefore F = \frac{1}{2}h(c_1 + c_2)$

$= \pi h(r_1 + r_2)$.

456. Corollary to 451 and 399.

The volume of the frustum of a circular cone,

$$V \cdot F = \tfrac{1}{4} a\pi (r_1^2 + 3r_3^2),$$

where r_3 is the radius of S.

457. Definition. A *circular cylinder* is the assemblage of straights each through a point of a given circle but not in its plane, and all parallel.

The portion of this assemblage included between two planes parallel to the circle is also called a circular cylinder. The sects the planes cut out are called the elements of the cylinder.

The two circles in these planes are called the *bases* of the cylinder.

The sect joining their centers is called the *axis*.

A sect perpendicular to the two planes is the *altitude* of the cylinder.

If the elements are perpendicular to the planes, it is a *right* cylinder; otherwise an *oblique* cylinder.

A section whose plane is perpendicular to the axis is called a *right section* of the cylinder. Any two elements, being equal and parallel, are opposite sides of a parallelogram; hence the bases and all sections parallel to them are equal circles.

A *truncated* cylinder is the portion between a base and a non-parallel section.

458. Theorem. *The volume of a circular cylinder is the product of its base by its altitude.*

Proof. If a prism and cylinder have equal altitudes and bases of equal area, any sections parallel to the bases are of equal area.

$\therefore$ (by 402) $V \cdot C = a\pi r^2$.

459. The lateral area of a circular cylinder is the product of an element by the length of a right section:

$$C = 2\pi ra.$$

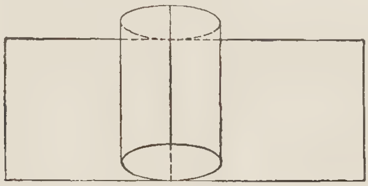

FIG. 168.

Proof. It is equal to the area of a parallelogram with one side an element and the consecutive side equal to the length of a base.

An altitude of this parallelogram equals the length of the right section.

460. Corollary to 459. The lateral area of a truncated circular cylinder is the product of the intercepted axis by the length of a right section.

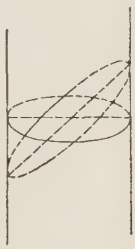

FIG. 169.

Proof. For substituting an oblique section for the right section through the same point of the axis changes neither the area nor the volume, since the portion between the sections is the same above as below either.

461. Corollary to 460. The volume of a truncated circular cylinder is the product of the intercepted axis by the area of the right section.

462. Archimedes' Theorem. *The volume of a sphere equals two-thirds the volume of the circumscribed cylinder.*

Proof. The volume of the circumscribed cylinder $= \pi r^2 \cdot 2r = 2\pi r^3$.

Ex. 582. In a right circular cylinder of altitude a, call the lateral area C and the area of the base B.

(1) Given a and C; find r.

(2) Given B and C; find a.

(3) Given C and $a = 2r$; find $C + 2B$.

(4) Given $C + 2B$ and $a = r$; find C.

(5) Given a and $B + C$; find r.

Ex. 583. The lateral area of a right circular cylinder is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

Ex. 584. In area, the bases of a right circular cylinder together are to the lateral surface as radius to altitude.

Ex. 585. If the altitude of a right circular cylinder is equal to the diameter of its base, the lateral area is four times that of the base.

Ex. 586. How much must the altitude of a right circular cylinder be prolonged to increase its lateral area by the area of a base?

Ex. 587. The lateral area of a right circular cone is twice the area of the base; find the vertical angle.

Ex. 588. Call the lateral area of a right circular cone K, its altitude a, the basal radius r, the slant height h.

(1) Given a and r; find K.

(2) Given a and h; find K.

(3) Given K and h; find r.

Ex. 589. How much canvas is required to make a conical tent 20 meters in diameter and 12 meters high?

Ex. 590. How far from the vertex is the cross-section which halves the lateral area of a right circular cone?

Ex. 591. Given the volume and lateral area of a right circular cylinder; find radius.

Ex. 592. Given lateral area and altitude of a right circular cylinder; find volume.

Ex. 593. A right cylinder of volume 50 has a circumference of 9; find lateral area.

Ex. 594. In a right circular cylinder of volume 8, the lateral area equals the sum of the bases; find altitude.

Ex. 595. If in three cylinders of the same height one radius is the sum of the other two, then one lateral area is the sum of the others, but contains a greater volume.

Ex. 596. What is the relation between the volumes of two cyliders when the radius of one equals the altitude of the other?

CHAPTER XV.

PURE SPHERICS.

463. If, instead of the plane and straight, we take the sphere and its great circle, that is, its geodesic or straightest, then much of our plane geometry holds good as spherics, and can be read off as spherics. Deducing spherics from a set of assumptions which give no parallels, no similar figures, we get a two-dimensional non-Euclidean geometry, yet one whose results are also part of three-dimensional Euclidean.

I. Assumptions of association on the sphere.

I 1′. For every point of the sphere there is always one and only one other point which with the first does not determine a straightest. This second point we will call the *opposite* of the first.

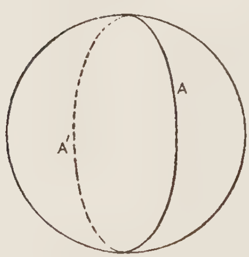

Fig. 170.

Two points, not each the other's opposite, always determine a straightest.

Such points are said to be *on* or of the straightest, and the straightest is said to be *through* them.

I 2′. Every straightest through a point is also through its opposite.

I 3′. Any two points of a straightest, not each the other's opposite, determine *this straightest;* and on every straightest there are at least two points not opposites.

I 4′. There are at least three points not on the same straightest.

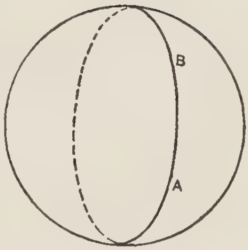

Fig. 171.

464. Theorem. *If O′ is the opposite of O, then O is the opposite of O′.*

Proof. If O is not the opposite of O', they determine a straightest. There is a point P not on this straightest (by I 4′), and this point is not the opposite of O, since it is not O'. $\therefore O, P$ determine (by I 1′) a straightest which (by I 2′) goes through O'. $\therefore O, O'$ do *not determine* a straightest.

465. Theorem. Two distinct straightests cannot have three points in common. [Proved as in 6.]

II. Assumptions of betweenness on the sphere.

466. These assumptions specify how "between" may be used of points in a straightest on a sphere.

II 1′. No point is between two opposites.

II 2′. No point is between its opposite and any third point.

II 3′. Between any two points not opposites there is always a third point.

II 4′. If B is between A and C, then B is also between C and A, and is neither C nor A.

II 5′. If A and B are not opposites, then there

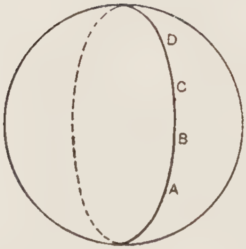

FIG. 172.

is always a point C such that B is between A and C.

II 6′. Of any three points, not more than one can be between the other two.

II 7′. If B is between A and C, and C is between A and D, then B is between A and D.

II 8′. Between no two points are there two opposites.

467. Definition. Two points A and B, not opposites, upon a straightest a, we call a *sect* and designate it with AB or BA. The points between A and B are said to be points *of* the sect AB or also situated *within* the sect AB. The remaining points of the straightest a are said to be situated without the sect AB. The points A, B are called *end-points* of the sect AB.

II 9′. (Pasch's assumption.) On the sphere, let A, B, C be three points, not all on a straightest, and no two opposites, and let a be a straightest on which are none of the points A, B, C; if then the straightest a goes through a point within the sect AB, it must always go either through a point of the sect BC or through a point of the sect AC; but it cannot go through both.

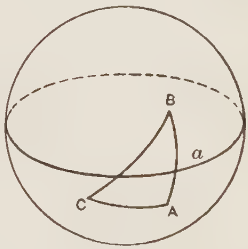

FIG. 173.

468. Theorem. Every straightest a separates

the other points of the sphere into two regions, of the following character: every point A of the one region determines with every point B of the other region, not its opposite, a sect AB within which lies a point of the straightest a; on the contrary, any two points A and A' of one and the same region always determine a sect AA' which contains no point of a.

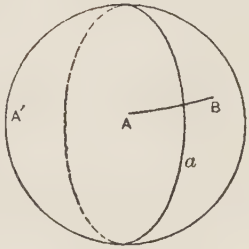

Fig. 174.

[Proved as in 22.]

Points in the same one of these two regions are said to be on the same *side* of a.

469. Theorem. The points of a straightest a other than two opposites, O, O', are separated by O, O' into two classes such that O or O' is between any point of the one and any non-opposite point of the other, but neither O nor O' is between two of the same class.

Proof. Take any other straightest b through O and $\therefore$ through O'. It (by 468) cuts the sphere into two regions. Now (by II 5′) a is not wholly in either of these regions; but all its points other than O and O' are in these regions. Two in the same region have no point of b between them. But O and O' are points of b. Two

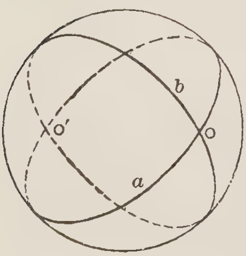

Fig. 175.

not opposites in different regions have a point of b between them; $\therefore$ either O or O'.

470. Definition. The parts of a straightest determined by a point of it O (with its opposite O') are called *rays* from O.

O and O' are called end-points of the rays.

II 10′. If C is a point of ray PP', every other point of the ray is between C and P or C and P'.

471. Theorem. Two opposites cannot both be on the same ray.

Proof. II 3′, II 10′ and II 2′.

472. Theorem. Every straightest has a point in common with any other.

Proof. If not, consider the straightest determined by any point of the one and a point of the other. This would have on one ray a pair of opposites, contrary to 471.

473. Definition. On the sphere, a system of sects, AB, BC, CD, . . . KL is called a *sect-train*, which joins the points A and L with one another.

The points within these sects together with their end-points are all together called the *points of the sect-train.*

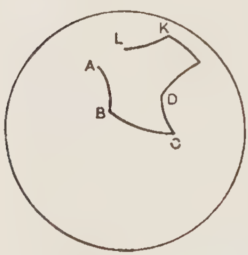

FIG. 176.

In particular, if the point L is identical with the point A, then the sect-train is called a *spherical polygon*. The sects are called the *sides* of the spherical polygon; their end-points its *vertices*.

Polygons with three vertices are called *spherical triangles.*

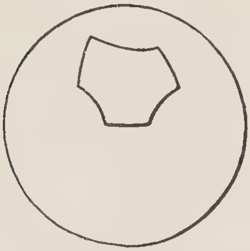

FIG. 177.

FIG. 178.

474. Theorem. Every spherical triangle separates the points of the sphere not pertaining to its sect-train into two regions, an *inner* and an *outer*. [As in 29.]

475. Convention. On a given straightest OA, the two rays OO', from O to its opposite O', are distinguished as of opposite *sense*. This distinction may be indicated by a qualitative use of the signs $+$ and $-$ (plus and minus), as in writing positive and negative numbers.

Any sect PO' or ray from P through O', or any sect PB where B is between P and O', has the sense of that ray OO' on which is P.

Then also BP is of sense opposite that of PB.

III. Assumptions of congruence on the sphere.

III 1′. If A, B are two points, not opposite, on a straightest a, and A' a point on the same or another straightest a', then we can find on a given ray of the straightest a' from A' always *one and*

only one point B', such that the sect AB is congruent to the sect $A'B'$.

Always $AB \equiv AB \equiv BA$.

III 2′. If $AB \equiv A'B'$ and $AB \equiv A''B''$, then is also $A'B' \equiv A''B''$.

III 3′. On the straightest a let AB and BC be two sects without common points, and furthermore $A'B'$ and $B'C'$ two sects on the same or another straightest, likewise without common points; if then $AB \equiv A'B'$ and $BC \equiv B'C'$, then is also $AC \equiv A'C'$.

476. Definition. On the sphere, let h, k be any two distinct rays from a point O, which pertain to *different* straightests. These two rays h, k from O we call a *spherical angle*, and designate it by $\widehat{\measuredangle}(h, k)$ or $\widehat{\measuredangle}(k, h)$.

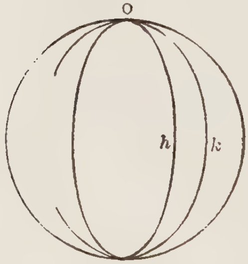

FIG. 179.

The rays h, k, together with the point O separate the other points of the sphere into two regions, the *interior* of the angle and the *exterior*. [As in 35.]

The rays h, k are called *sides* of the angle, and the point O the *vertex*.

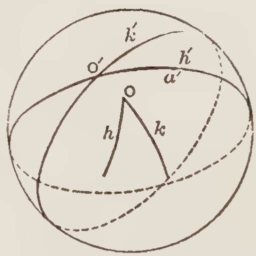

FIG. 180.

III 4′. On the sphere, given a spherical angle $\widehat{\measuredangle}(h, k)$, and a straightest a', also a determined side of a'. Designate by h' a ray of the straightest a' starting from the point O': then is there *one and only one* ray k' such that the

$\widehat{\sphericalangle}(h, k)$ is congruent to the angle $\widehat{\sphericalangle}(h', k')$, and likewise all inner points of the angle $\widehat{\sphericalangle}(h', k')$ lie on the given side of a'.

Always $\widehat{\sphericalangle}(h, k) \equiv \widehat{\sphericalangle}(h, k) \equiv \widehat{\sphericalangle}(k, h)$.

III 5′. If $\widehat{\sphericalangle}(h, k) \equiv \widehat{\sphericalangle}(h', k')$ and $\widehat{\sphericalangle}(h, k) \equiv \widehat{\sphericalangle}(h'', k'')$, then is also $\widehat{\sphericalangle}(h', k') \equiv \widehat{\sphericalangle}(h'', k'')$.

477. Convention. On the sphere let ABC be an assigned spherical triangle; we designate the two rays going out from A through B and C by h and k respectively. Then the angle $\widehat{\sphericalangle}(h, k)$ is called the angle of the triangle ABC included by the sides AB and AC, or opposite the side BC.

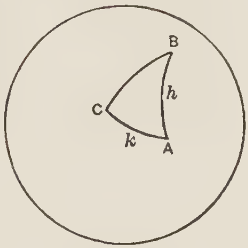

FIG. 181.

It contains in its interior all the inner points of the spherical triangle ABC and is designated by $\widehat{\sphericalangle} BAC$ or $\widehat{\sphericalangle} A$.

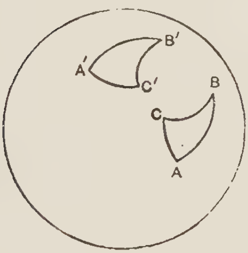

FIG. 182.

III 6′. On the sphere, if for two triangles ABC and $A'B'C'$ we have the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\widehat{\sphericalangle} BAC \equiv \widehat{\sphericalangle} B'A'C'$, then also always are fulfilled the congruences $\widehat{\sphericalangle} ABC \equiv \widehat{\sphericalangle} A'B'C'$ and $\widehat{\sphericalangle} ACB \equiv \widehat{\sphericalangle} A'C'B'$.

478. Convention. When the sect AB is set off on a ray starting from A, if the point B falls within the sect AC, then the sect AB is said to be *less* than the sect AC.

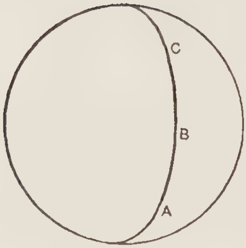

FIG. 183.

In symbols, $AB < AC$.

Then also AC is said to be *greater* than AB.

In symbols, $AC > AB$.

$AB > CD$ when E between A and B gives $AE = CD$ or $BE = CD$, using $=$ for $\equiv$.

479. Convention. When $\sphericalangle AOB$ is set off from vertex O' against one of the rays of $\sphericalangle A'O'C$ toward the other ray, if its second side falls within $\sphericalangle A'O'C$, then the $\sphericalangle AOB$ is said to be *less* than the $\sphericalangle A'O'C$.

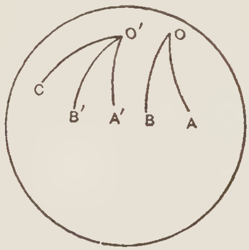

FIG. 184.

In symbols,

$$\sphericalangle AOB < \sphericalangle A'O'C.$$

Then also $\sphericalangle A'O'C$ is said to be greater than $\sphericalangle AOB$.

In symbols, $\sphericalangle A'O'C > \sphericalangle AOB$.

FIG. 185.

480. Definition. Two spherical angles, which have the vertex and one side in common and whose not-common sides make a straightest are called *adjacent angles*.

481. Definition. Two spherical angles with a common vertex and whose sides make two straightests are called *vertical angles.*

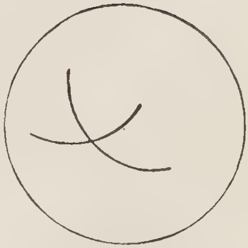

Fig. 186.

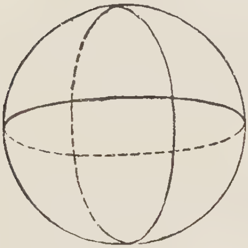

Fig. 187.

482. Definition. A spherical angle which is congruent to one of its adjacent angles is called a *right angle.*

Two straightests which make a right angle are said to be *perpendicular* to one another.

483. Convention. If A, B are points which determine a straightest, then we may designate one of the regions or hemispheres it makes as *right* from the straightest AB taken in the sense of the sect AB (and the same hemisphere as *left* from BA taken in the sense from B to A).

If now C is any point in the right hemisphere from AB, then we designate that hemisphere of AC in which B lies as the left hemisphere of AC. So we can finally fix for each straightest which hemisphere is right from this straightest taken in a given sense.

Of the sides of any angle, that is designated as the right which lies on the right hemisphere of

that straightest which is determined (also in sense) by the other side, while the left side is that lying on the left of the straightest which is determined (also in sense) by the other side.

Two spherical triangles with all their sides and angles respectively congruent are called *congruent* if the right side of one angle is congruent to the right side of the congruent angle, and its left side to that angle's left; but if the right side of one angle be congruent to the left side of the congruent angle, and its left side to that angle's right, the triangles are called *symmetric*.

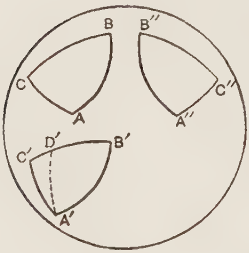

Fig. 188.

484. Theorem. Two spherical triangles are either congruent or symmetric if they have two sides and the included angle congruent.

[Proved as in 43.]

485. Theorem. Two spherical triangles are either congruent or symmetric if a side and the two adjoining angles are respectively congruent.

[Proved as in 44.]

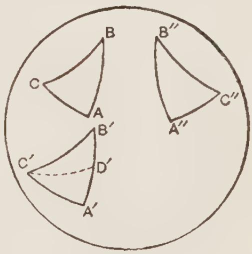

Fig. 189.

486. Theorem. If two spherical angles are congruent, so are also their adjacent angles.

[Proved as in 45.]

487. Theorem. Vertical spherical angles are congruent.

[Proved as in 46.]

488. Theorem. All right angles are congruent.

[Proved as in 50.]

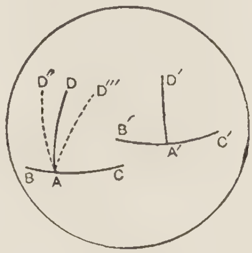

FIG. 190.

489. Theorem. At a point A of a straightest a there is not more than one perpendicular to a.

[Proved as in 52.]

490. Definition. When any two spherical angles are congruent to two adjacent spherical angles each is said to be the *supplement* of the other.

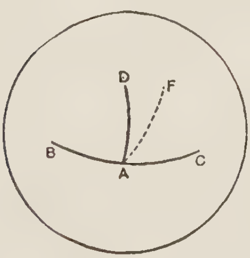

FIG. 191.

491. Definition. If a spherical angle can be set off against one of the rays of a right angle so that its second side lies within the right angle, it is called an *acute* angle.

492. Definition. A spherical angle neither right nor acute is called an obtuse angle.

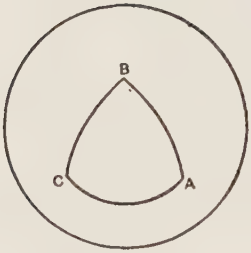

FIG. 192.

493. Definition. A spherical triangle with two sides congruent is called *isosceles.*

494. Theorem. The angles opposite the congruent sides of an isosceles triangle are congruent. [Proved as in 57.]

495. Theorem. If two angles of a spherical triangle be congruent, it is isosceles.

[Proved as in 485.]

496. Theorem. Two spherical triangles are either congruent or symmetric if the three sides of the one are congruent, respectively, to the three sides of the other.

[Proved as in 58.]

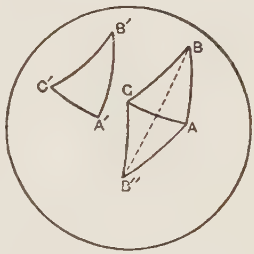

FIG. 193.

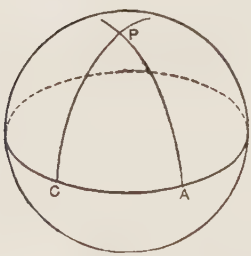

FIG. 194.

497. Theorem. *Any two straightests perpendicular to a given straightest intersect in a point from which all sects to the given straightest are perpendicular to it and congruent.*

Given r't $\widehat{\measuredangle} A \equiv \widehat{\measuredangle} C$.

To prove $PA \equiv PC \equiv PD$, and $\widehat{\measuredangle} D$ r t.

Proof. By 495, $PA \equiv PC$ and (by 485) $PA \equiv P'A$.

$\therefore$ (by 484) $\widehat{\measuredangle} PDA \equiv \widehat{\measuredangle} P'DA$;

$\therefore$ (by 488) $\widehat{\measuredangle} PDA \equiv \widehat{\measuredangle} PAD$;

$\therefore$ (by 495) $PD \equiv PA$.

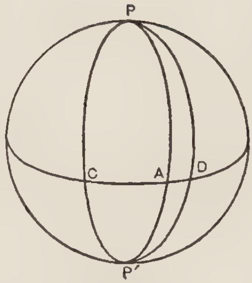

FIG. 195.

498. Definition. The two opposite points at which two perpendiculars to a given straightest intersect are called its *poles*, and it the *polar* of either pole.

A sect from a pole to its polar is called a *quadrant*.

499. Theorem. *All quadrants are congruent.*

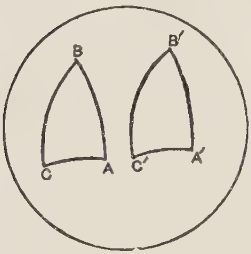

FIG. 196.

Let AB and $A'B'$ be two quadrants.

To prove $AB \equiv A'B'$.

Proof. At A take a r't $\widehat{\measuredangle}$ BAC, and also at A'. On AC take a sect AC, and on $A'C'$ take $A'C' \equiv AC$. At C and C' take straightests $\perp AC$ and $A'C'$. These contain B and B'. $\therefore$ (by 485) $AB \equiv A'B'$.

500. Theorem. *A point which is a quadrant from two points of a straightest not opposites is its pole.*

Let PA and PC be two quadrants.

Proof, At A and C erect

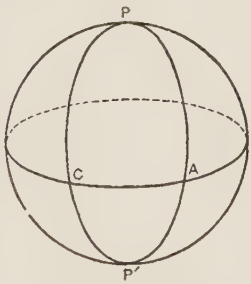

FIG. 197.

perpendiculars intersecting at P'. Then (by 499 and 496) $\widehat{\measuredangle} PAC \equiv \widehat{\measuredangle} P'AC$.

501. Theorem. *If three sects from a point to a straightest be equal, they are quadrants.*

Proof. They are sides of two adjacent isosceles triangles, and hence perpendiculars.

502. Contranominal of 501. If three equal sects from a point be not quadrants, their three other end-points are not on a straightest.

503. Theorem. *Through a point A, not on a straightest a, there is to a always a perpendicular straightest which, if A be not a pole of a, is unique.*

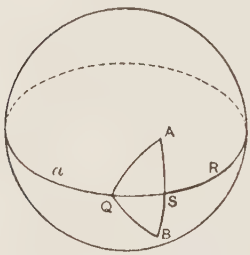

Fig. 198.

Proof. Take any sect QR on a. Take on the other side of a from A, $\widehat{\measuredangle} BQR \equiv \widehat{\measuredangle} AQR$, and $QB \equiv QA$.

Then (by 484) $AB \perp a$ at S.

Moreover, if there were a second straightest perpendicular to a from A, then A would (by 498) be a pole of a.

504. Definition. A point B of a given ray OO' such that $BO \equiv BO'$ will be called the bisection-point of the ray. A point B between A and C such that $AB \equiv BC$ is called the bisection-point of the sect AC.

505. Problem. *To bisect a given ray OO'.*

Construction. At two points of the given ray not

both end-points erect perpendiculars [take (by III 4′) angles ≡ to $\widehat{\measuredangle} S$ in 503], intersecting at P. Take another ray from O, not on the same straightest as the given ray, and at two points of it not both end-points erect perpendiculars intersecting at Q. The straightest PQ bisects the given ray OO'.

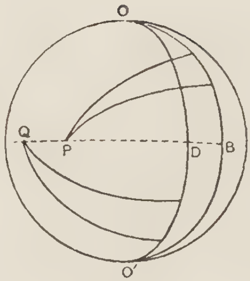

FIG. 199.

Proof. Since P and Q are poles, ∴ $\widehat{\measuredangle} B \equiv$ r't $\widehat{\measuredangle} \equiv \widehat{\measuredangle} D$. ∴ (by 485) $OB = BO'$.

506. Theorem. *If O and O' are opposites, then with vertex O $\widehat{\measuredangle}$ $(h, k) \equiv \widehat{\measuredangle}$ (h, k) with vertex O'.*

Proof. Bisect (by 505) ray h at A and ray k at C. Then (by 496) $\widehat{\measuredangle} AOC \equiv \widehat{\measuredangle} AO'C$.

507. Definition. From the vertex O, a ray b within $\widehat{\measuredangle}(h, k)$ making $\widehat{\measuredangle}(h, b) \equiv \widehat{\measuredangle}(b, k)$ will be called the bisector of $\measuredangle(h, k)$.

508. Problem. *To bisect a given spherical angle.*

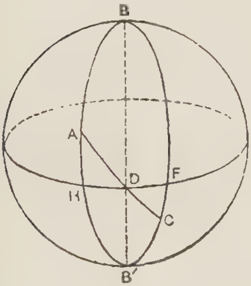

FIG. 200.

Construction. By 505, bisect the rays of the angle $\widehat{\measuredangle} B$ at H and F. Take A between H and B and from F on FB' take $FC \equiv HA$. Then AC intersects HF at D, and BD bisects $\widehat{\measuredangle} HBF$.

Proof. By 496, $\widehat{\measuredangle} ACB \equiv \widehat{\measuredangle} CAB'$; ∴ by 485, $HD = FD$; ∴ by 496, $\measuredangle HBD \equiv \widehat{\measuredangle} FBD$.

509. Problem. *To bisect a given sect.*

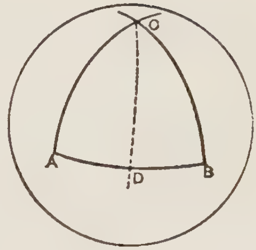

FIG. 201.

Construction. At the end-points erect perpendiculars [by taking (by III 4′) angles $\equiv \widehat{\measuredangle} S$ in 503].

Bisect (by 508) the $\widehat{\measuredangle}$ between them.

Proof. By 497 and 484.

510. Corollary. In an isosceles triangle the bisector of the angle between equal sides bisects at right angles the third side.

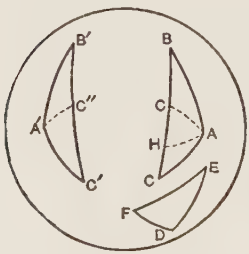

FIG. 202.

511. Theorem. If two spherical triangles have two sides of the one equal respectively to two sides of the other, and the angles opposite one pair of equal sides equal, then the angles opposite the other pair are either equal or supplemental.

[Proved as in 175.]

512. Definition. In any spherical triangle the sect having as end-points a vertex and the bisection-point of the opposite side is called a *median*.

513. Theorem. *An angle adjacent to an angle of a spherical triangle is greater than, equal to, or less than either of the interior non-adjacent angles, according as the median from the other interior non-adjacent angle is less than, equal to, or greater than a quadrant. And inversely.*

Proof. Let $\widehat{\measuredangle} ACD$ be an angle adjacent to $\widehat{\measuredangle} ACB$

of $\widehat{\triangle}ACB$. Bisect AC at F. On straightest BF beyond F take $FH \equiv FB$. $\therefore$ (by 484) $\widehat{\measuredangle}BAF \equiv \widehat{\measuredangle}HCF$. If now the median BF be a quadrant BFH is a ray and H is on BCD. If the median BF be less than a quadrant, H' is within $\widehat{\measuredangle}ACD$.

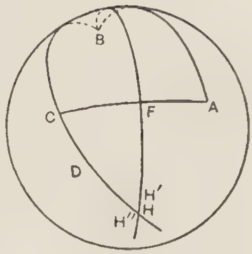

FIG. 203.

$\therefore \widehat{\measuredangle}H'CA < \widehat{\measuredangle}DCA. \qquad \therefore \widehat{\measuredangle}DCA > \widehat{\measuredangle}BAC.$

If BC be greater than a quadrant, H'' is without $\widehat{\measuredangle}ACD$.

$\therefore \widehat{\measuredangle}H''CF > \widehat{\measuredangle}DCF. \qquad \therefore \widehat{\measuredangle}DCA < \widehat{\measuredangle}BAC.$

514. Definition. Two sects respectively congruent to two made by a point on a ray with its end-points are called *supplemental*.

515. Theorem. The supplements of congruent sects are congruent.

Proof. They are sums or differences of quadrants and congruent sects less than quadrants; and (by 499) all quadrants are congruent.

516. Theorem. *If a median be a quadrant, it is an angle-bisector, and the sides of the bisected angle are supplemental.*

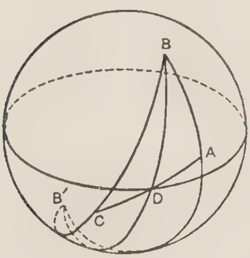

FIG. 204.

Let median BD in $\widehat{\triangle}ABC$ be a quadrant.

Proof. In $\widehat{\triangle}ABD$ and $\widehat{\triangle}CB'D$ (by 484) $AB = CB'$ and $\widehat{\measuredangle}ABD \equiv \widehat{\measuredangle}CB'D \equiv \widehat{\measuredangle}CBD$ (by 506).

517. Inverse. If two sides of a triangle are supplemental, the median is a quadrant.

Proof. $\widehat{\triangle} ABC \equiv$ or $\dotplus \widehat{\triangle} AB'C$ (by 496).

$\therefore \widehat{\triangle} ABD \equiv$ or $\dotplus \widehat{\triangle} CB'D$ (by 484). $\therefore BD = DB'$.

518. Corollary. If two sides of a triangle are supplemental, the opposite angles are supplemental.

519. Theorem. *Two spherical triangles are either congruent or symmetric if they have two angles of the one respectively equal to two of the other, the sides opposite one pair equal, and those opposite the other pair not supplemental.*

Given $\widehat{\measuredangle} B \equiv \widehat{\measuredangle} E$; $\widehat{\measuredangle} C \equiv \widehat{\measuredangle} F$; $AB \equiv DE$.; AC not supplemental to FD.

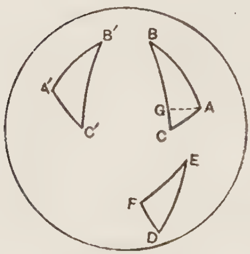

Fig. 205.

Proof. On ray BC take $BG = EF$. G must be C, else would we have a $\widehat{\triangle} ACG$ with adjacent $\widehat{\measuredangle} AGB \equiv \widehat{\measuredangle} ACG$ interior non-adjacent and $\therefore$ with median a quadrant (by 513) and $\therefore$ (by 516) with AC supplemental to AG, that is, to FD.

520. Theorem. *Two spherical triangles are either congruent or symmetric if they have in each one, and only one, right angle, equal hypothenuses and another side or angle congruent.*

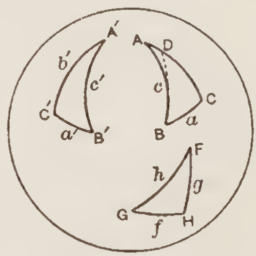

Fig. 206.

Given $\widehat{\measuredangle} C \equiv \widehat{\measuredangle} H \equiv$ r't $\widehat{\measuredangle}$, and $c = h$. If $a = f$, then if $AC > g$, take $CD = g$. $\therefore BD = h = c$, and (by 510) the

bisector of $\widehat{\measuredangle} DBA$ is $\perp$ to CDA. $\therefore$ (by 498) B is pole to CDA. $\therefore \widehat{\measuredangle} A$ is also r't.

If $\widehat{\measuredangle} A \equiv \widehat{\measuredangle} F$, then if $\widehat{\measuredangle} ABC > \widehat{\measuredangle} G$, take $\widehat{\measuredangle} ABD \equiv \widehat{\measuredangle} G$. $\therefore$ (by 485) $\widehat{\measuredangle} BDA \equiv \widehat{\measuredangle} H \equiv \widehat{\measuredangle} C \equiv$ r't $\widehat{\measuredangle}$. $\therefore B$ is pole to CDA.

521. Theorem. *The straightest through the poles of two straightests is the polar of their intersection-points.*

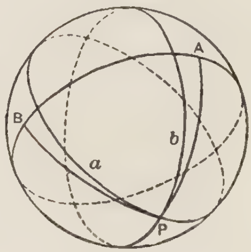

FIG. 207.

Let A and B be poles of a and b, which intersect in P.

To prove AB the polar of P.

Proof. AP and BP are quadrants.

522. Corollary to 521. The straightest through the poles of two straightests is perpendicular to both.

523. Corollary to 521. If three straightests are copunctal, their poles are on a straightest.

524. Definition. If A, B, C are the vertices and a, b, c the opposite sides of a spherical triangle, and A' that pole of a on the same side of a as A, B' of b as B, C' of c as C, then $A'B'C'$ is called the *polar triangle* of ABC.

525. Definition. Of a spherical triangle A, B, C, the *polar triangle* is A', B', C' where A' is that pole of BC or a on the same side of a as A, B' of b as B, C' of c as C.

526. Theorem. *If of two spherical triangles the second is the polar of the first, then the first is the polar of the second.*

Let ABC be the polar of $A'B'C'$.

To prove $A'B'C'$ the polar of ABC.

Proof. Since B is pole of $A'C'$, $\therefore$ BA' is a quadrant; and since C is pole of $A'B'$, $\therefore$ CA' is a quadrant; $\therefore$ (by 500) A' is pole of BC. In like manner, B' is pole of AC, and C' of AB. Moreover, since by hypothesis A and A' are on the same side of $B'C'$ and A is pole of $B'C'$, $\therefore$ sect AA' is less than a quadrant. $\therefore$ A and A' are on the same side of BC, of which A' is pole. And so for B' and C'.

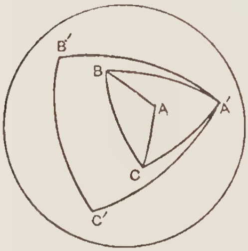

FIG. 208.

527. Theorem. *In a pair of polar triangles, any angle of either intercepts, on the side of the other which lies opposite it, a sect which is the supplement of that side.*

Let ABC and $A'B'C'$ be two polar triangles.

Proof. Call D and E respectively the points where ray $A'B'$ and ray $A'C'$ meet BC. Since B is pole of $A'C'$, $\therefore$ BE is a quadrant, and since C is pole of $A'B'$, $\therefore$ CD is a quadrant.

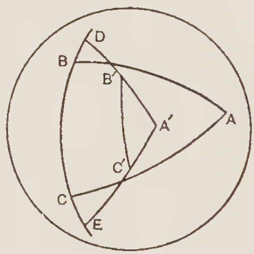

FIG. 209.

$$\text{But } BE+CD=BC+CE+CB+BD=$$
$$BC+(EC+CB+BD)=BC+DE.$$

528. Theorem. *Two spherical triangles are either congruent or symmetric if they have three angles of the one respectively equal to three angles of the other.*

Proof. Since the given triangles are respectively equiangular, their polars are respectively equilateral. For (by 484) equal angles at the poles of straightests intercept equal sects on those straightests; and these equal sects are the supplements of corresponding sides of the polars Hence these polars, having three sides respectively equal, are respectively equiangular. Therefore the original triangles are respectively equilateral, which was to be proved.

529. Corollary to 511. Two spherical triangles are either congruent or symmetric if they have two sides of the one respectively equal to two of the other, the angles opposite one pair equal, and those opposite the other pair not supplemental.

530. Theorem. *If two sides of a spherical triangle are each less than a quadrant, any sect from the third side to the opposite vertex is less than a quadrant.*

Let AB and BC be each less than a quadrant.

To prove $BD <$ quadrant.

Proof. Let FG, the polar of B, meet BD at H. If H were between B and D, then GHF would (by II 9′) meet CA, and so have a point on each of the three sides of $\widehat{\triangle} AB'C$, which (by II 9′) is impossible. Hence D is between B and H. That is $BD <$ quadrant.

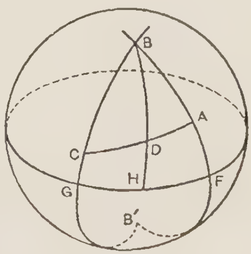

FIG. 210.

531. Corollary to 530 and 513. If two sides of a spherical triangle are each less than a quadrant, the

angle opposite either is less than the supplement of the angle opposite the other.

532. Theorem. *If two sides of a spherical triangle be each less than a quadrant, as the third side is greater or less than one of these, so is it with the opposite angles. And inversely.*

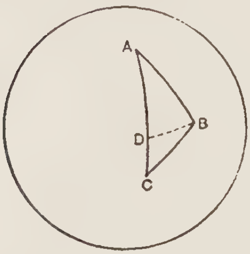

FIG. 211.

Let in $\widehat{\triangle}ABC$, BC and another side, AB, be each less than a quadrant, and $AC > AB$.

To prove $\widehat{\measuredangle}ABC > \widehat{\measuredangle}ACB$.

Proof. Within AC take D making $AD = AB$. Then (by 530) DB is less than a quadrant. $\therefore$ (by 531) $\widehat{\measuredangle}ADB > \widehat{\measuredangle}C$. But $\widehat{\measuredangle}ABC > \widehat{\measuredangle}ABD \equiv \widehat{\measuredangle}ADB > \widehat{\measuredangle}C$.

533. Theorem. *If the three sides of a spherical triangle are each less than a quadrant, any two are together greater than the third.*

[Proved as in 174.]

534. Definition. On the sphere, the assemblage of points which with a given point give congruent sects is called a circle. The given point is called a pole of the circle. Any one of the congruent sects is called a *spherical radius* of the circle.

Thus a straightest is a circle with a quadrant for spherical radius. But henceforth, for convenience, by circle we will mean a circle with a radius not a quadrant.

A sect whose end-points are on a circle is called a *spherical chord*, or simply a chord.

A chord containing a pole is called a diameter.

Since the supplements of congruent sects are (by

515) congruent, therefore every circle has two poles which are opposite points, and its spherical radius to one pole is the supplement of that to the other.

Always one spherical radius is less than a quadrant. Call its pole the q-pole, and it the q-radius.

535. Theorem. *Any spherical chord is bisected by the perpendicular from a pole.*

Proof. $AD = BD$ (by 520).

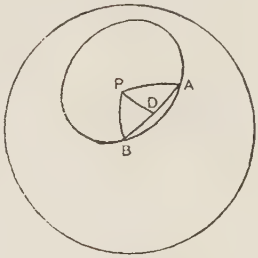

FIG. 212.

536. Corollary. A straightest perpendicular to a diameter at an end-point has only this point in common with the circle.

537. Definition. A straightest with one and only one point in common with a circle is called a tangent to the circle.

538. Theorem. *If an oblique from a point to a straightest be less than a quadrant, then there is one and only one perpendicular sect from the point to the straightest which meets it at less than a quadrant from the foot of the oblique and this is less than a quadrant.*

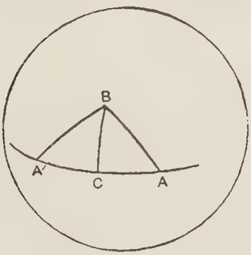

FIG. 213.

Let BA be oblique to CA and $<q$, and $BC \perp CA$.

Proof. Then CA cannot $=q$, else would $BA = q$. Hence CA may be taken $<q$, since from C to its opposite $=2q$. Now take $CA' = CA$. Then $BA' = BA$ and BC is median where the two sides are each $<q$. $\therefore$ (by 530) $BC < q$. $\therefore$ (by 503) its prolongation BC' is the only other $\perp$ from B to AC.

539. Definition. If A be a point of a circle whose q-pole is P, then P or any point between A and P is said to be within the circle, while Q such that A is between P and Q is said to be without the circle.

540. Theorem. *Any straightest through an end-point of a diameter, but not perpendicular to the diameter, has a point within and a second point on the circle.*

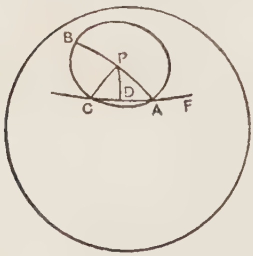

FIG. 214.

Let P be the q-pole, and AC the straightest through A, an end-point of diameter BPA ($\widehat{\measuredangle} PAC$ not r't).

Proof. Take (538) $PD \perp CA$ with PD and AD each $<q$. Take $DC = DA$.

$\therefore$ (by 484) $PC = PA$; that is, C on circle. Moreover (by 513) $\widehat{\measuredangle} PAF > \widehat{\measuredangle} PCA \equiv \widehat{\measuredangle} PAC$. $\therefore \widehat{\measuredangle} PAD$ acute. $\therefore$ (by 532) $PD < PA$; that is, D within circle.

541. Theorem. *Any straightest with a point on and a point within a circle has a second point on the circle.*

Let AB have a point A on and B within circle with q-pole P.

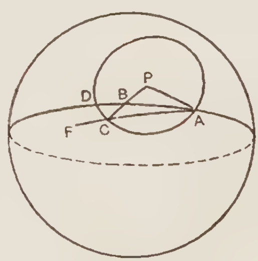

FIG. 215.

Proof. $\widehat{\measuredangle} PAB$ cannot be r't. For if so, then producing PB to meet the circle at C, (by 531) $\widehat{\measuredangle} PCF > \widehat{\measuredangle} PAC$ $>$ r't $\widehat{\measuredangle} PAB$. $\therefore \widehat{\measuredangle} PCA$ adjacent to obtuse $\widehat{\measuredangle} PCF$ is acute. But it is also obtuse, being (by 494)

$\equiv \hat{\measuredangle} PAC$. This is impossible, $\therefore BA$ not $\perp AP$; $\therefore$ (by 540) it has a second point on the circle.

542. Corollary. A tangent has no point within the circle.

543. Theorem. *If less than a quadrant, the perpendicular is the least sect from a point to a straightest.*

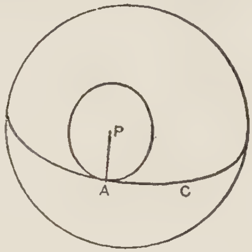

FIG. 216.

Proof. If any other sect from P to AC were less than the perpendicular PA, then AC would have a point within the circle with q-pole P and q-radius PA, and $\therefore$ (by 541) a second point on this circle, which (by 536) is impossible.

544. Convention. In general a sum of sects is a number of quadrants plus a sect.

545. Theorem. *Any two sides of a spherical triangle are together greater than the third.*

Proof. Since each side is less than two quadrants, we have only to prove $AB + BC > AC$ when $AB < q$, and $BC < AC$.

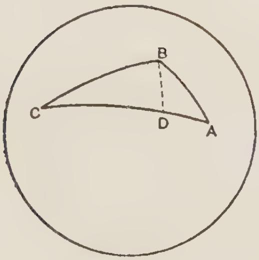

FIG. 217.

I. If $BC = q$, then taking $CD = q$, we have (by 500) $BD \perp AC$.

$\therefore$ (by 543) $AD < AB$.

$\therefore \quad AC = AD + DC < AB + BC$.

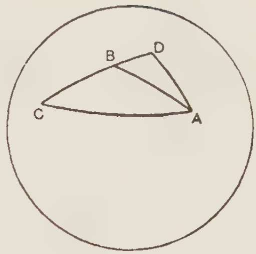

FIG. 218.

II. If $BC<q$,

(1) if $CA<q$, this is 533.

(2) if $CA=q$, erect $\perp$ at A.

$\therefore$ (by 543) $AB>BD$.

$\therefore$ $AB+BC>DB+BC=DC$
$=AC$.

(3) If $CA>q$, take on it $CD=q$ and make $CBF=q$. Then sects AB and DF cross at K, for F is on the non-C-side of AB, while D is on the C-side of AB. $\therefore DF$ must have a point on straight AB. But all points of sect DF are interior to $\widehat{\sphericalangle}C$, $\therefore$ this intersection point is on sect AB, which is all of straight AB within $\widehat{\sphericalangle}C$.

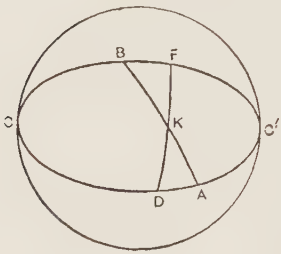

FIG. 219.

Then (by 543) $BF<BK$ and $AD<AK$.

$\therefore AC=AD+DC<AK+CD$
$=AK+CF<CB+BK+KA$.

III. If $BC>q$, then in ABC' all sides are less than quadrants.

$\therefore$ (by 533) $CB+BA+AC'>CB+BC'=CA+AC'$.

$\therefore CB+BA>CA$.

546. Definition. A convex spherical polygon is one no points of which are on different sides of the straightest of any of its sides.

547. Theorem. *A convex spherical polygon is less than one containing it.*

548. Theorem. *The sum of the sides of a convex spherical polygon is less than four quadrants.*

Proof. It is within, hence less than, any one of its angles.

549. Theorem. *If one angle of a spherical triangle be greater than a second, the side opposite the first must be greater than the side opposite the second; and inversely.*

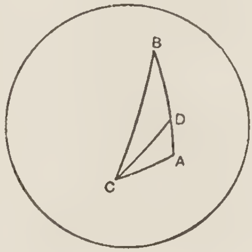

FIG. 220.

Given $\widehat{\measuredangle} C > \widehat{\measuredangle} B$.

Proof. Take $\widehat{\measuredangle} DCB \equiv \widehat{\measuredangle} B$. Then (by 495) $DC = DB$. But (by 545) $DC + DA > AC$.

550. Theorem. *In a cyclic quadrilateral, the sum of one pair of opposite angles equals the sum of the other pair.*

Proof. By isosceles triangles.

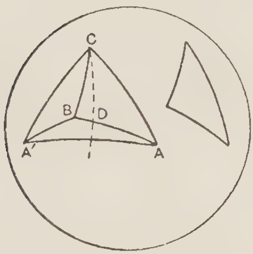

FIG. 221.

551. Theorem. *Of sects joining two symmetrical points to a third, that cutting the axis is the greater.*

Proof. $BA = BD + DA$
$= BD + DA' > BA'$.

552. Theorem. *If two spherical triangles have two sides of the one equal to two sides of the other, but the included angles unequal, then that third side is the greater which is opposite the greater angle; and inversely.*

Proof. Against one of the equal sides of one triangle construct a triangle with elements equal to those in the other. Bisect the angle made by the pair of equal sides. This axis cuts the third side, which is opposite the greater angle.

553. Theorem. *If each of the two sides about a right angle is less than a quadrant, then the hypothenuse is less than a quadrant.*

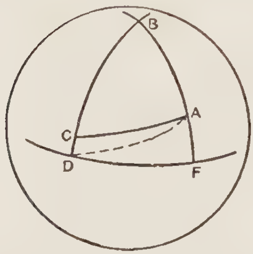

Fig. 222.

Proof. Extend the two sides BA, BC, taking $BF = BD =$ quadrant. Then (by 500) B is pole of DF. ∴ (by 498 and 497) $\widehat{\measuredangle} F$ is r't. ∴ (by 495) DF is a quadrant. ∴ (by 500) DA is a quadrant. ∴ (by 530) $AC <$ quadrant.

554. Inverse of 553.

If the hypothenuse and a side are each less than a quadrant, then the other side is less than a quadrant.

Proof. If B is r't (Fig. 222), and AB and AC each $<q$, there is (by 538) on st' AB a p't H such that CH and AH each $<q$ while $CH \perp AH$.

But H is B or B'.

It cannot be B' since $BA < q$ and ∴ (by II 10′) $AB' > q$.

555. Theorem. *The straightest bisecting two sides of a triangle meets the third side at a quadrant from its bisection-point.*

Let the straightest through A', B', the bisection-points of two sides BC, CA, meet the third side produced at D and D'.

Proof. Take (by 538) AL, BM, $CN \perp A'B'$ and such that each is $<q$, and *also* $B'L$, $B'N$, $A'M$, $A'N$ each $<q$.

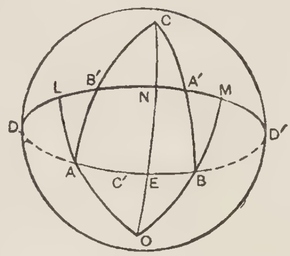

FIG. 223.

$\therefore$ in $\widehat{\triangle}$'s ALB' and CNB' (by 519) $AL = CN$. Similarly $BM = CN$. $\therefore$ in $\widehat{\triangle}$'s ALD and BMD' (by 519) $AD = BD'$. $\therefore$ if C' be bisection-point of AB, we have $C'A + AD = C'B + BD' = q$.

556. Theorem. The end-points of any sect taken with any point on its perpendicular bisector give equal sects.

557. Corollary 556. Every point on the perpendicular bisector of a sect is pole of a circle through its end-points.

558. Corollary to 557.

The perpendicular bisectors of the sides of a spherical triangle are copunctal (in its *circumcenter*).

559. Corollary I to 555.

The altitudes of a spherical triangle are copunctal (in its *orthocenter*.

For, regarding $A'B'C'$ as the triangle, the perpendicular to DC' at C' is the polar of D, and $\therefore \perp$ to $A'B'$.

Similarly, the perpendicular to BA' at A' is $\perp$ to $B'C'$, etc.

So the three altitudes of $A'B'C'$ are copunctal in the circumcenter of ABC.

560. Corollary II to 555. (Lexell).

The vertices of spherical triangles of the same

angle-sum on the same base are on a circle copolar with the straightest bisecting their sides.

For $AO = BO$, $\widehat{\measuredangle} OAB = \widehat{\measuredangle} OBA$, $\widehat{\measuredangle} LAB = \widehat{\measuredangle} MBA = \frac{1}{2}[A + B + C]$. Hence $\widehat{\triangle} AOB$ is fixed, and $\therefore OC$ [supplemental to OA].

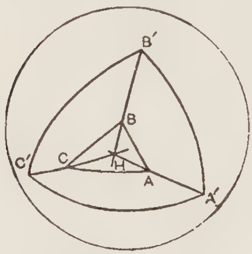

FIG. 224.

561. Theorem. *The straightests through the corresponding vertices of a triangle and its polar are copunctal in the common orthocenter.*

Proof. For AA' is $\perp$ to BC and $B'C'$, since it passes through their poles.

Equivalence.

562. Theorem. *Any angle made with a side of a spherical triangle by joining its end-point to the circumcenter, equals half the angle-sum less the opposite angle of the triangle.*

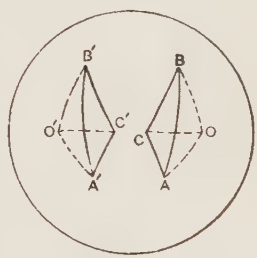

FIG. 225.

Proof. For $\widehat{\measuredangle} A + \widehat{\measuredangle} B + \widehat{\measuredangle} C = 2\,\widehat{\measuredangle}\,OCA + 2\,\widehat{\measuredangle}\,OCB \pm 2\widehat{\measuredangle} OAB$. $\therefore \widehat{\measuredangle} OCA = \frac{1}{2}[\widehat{\measuredangle} A + \widehat{\measuredangle} B + \widehat{\measuredangle} C] - [\widehat{\measuredangle} OCB \pm \widehat{\measuredangle} OAB] = \frac{1}{2}[\widehat{\measuredangle} A + \widehat{\measuredangle} B + \widehat{\measuredangle} C] - \widehat{\measuredangle} B$.

563. Corollary to 562. Symmetrical spherical triangles are equivalent or equivalent by completion.

For the three pairs of isosceles triangles formed by joining the vertices to the circumcenters, hav-

ing respectively a side and two adjoining angles congruent, are congruent.

564. Theorem. *Of the triangles formed by three non-copunctal straightests, two containing vertical angles are together equivalent to that angle.*

To prove $\widehat{\triangle} ABC + \widehat{\triangle} AB'C' = \widehat{\sphericalangle} ABA'CA$.

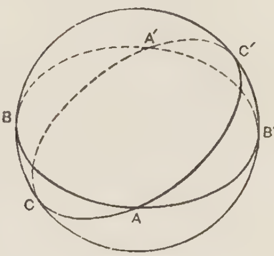

FIG. 226.

Proof. $B'C' = BC$, each being supplement of CB'. Again $AC' = A'C$ (supplements of AC). Again $AB' = A'B$ (supplements of AB). $\therefore$ (by 496) $\widehat{\triangle} AB'C' = \widehat{\triangle} BCA'$.

$\therefore \widehat{\triangle} ABC + \widehat{\triangle} AB'C' = \widehat{\triangle} ABC + \widehat{\triangle} BCA' = \widehat{\sphericalangle} ABA'CA$.

565. The *spherical excess*, *e*, of a spherical triangle is the excess of the sum of its angles over two right angles.

In general the spherical excess of a spherical polygon is the excess of the sum of its angles over twice as many right angles as it has sides less two.

566. Theorem. *A spherical triangle is equivalent to half its spherical excess.*

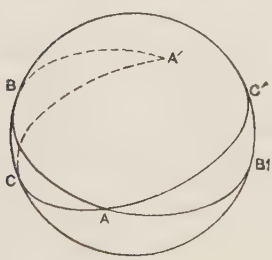

FIG. 227.

Proof. Produce the sides of the $\widehat{\triangle} ABC$ until they meet again two and two at A', B', C'. The $\widehat{\triangle} ABC$ now appears in three angles, $\widehat{\sphericalangle} A$, $\widehat{\sphericalangle} B$, $\widehat{\sphericalangle} C$. But (by 564) $\widehat{\sphericalangle} A = \widehat{\triangle} ABC + \widehat{\triangle} AB'C'$. $\therefore \widehat{\sphericalangle} A + \widehat{\sphericalangle} B + \widehat{\sphericalangle} C = 2$ r't $\widehat{\sphericalangle}$'s $+ 2 \widehat{\triangle} ABC$.

$\therefore 2 \widehat{\triangle} ABC = \widehat{\sphericalangle} A + \widehat{\sphericalangle} B + \widehat{\sphericalangle} C - 2$ r't $\widehat{\sphericalangle}$'s $= e$.

567. Corollary I to 566. The sum of the angles of a $\overset{\frown}{\triangle}$ is >2 r't $\overset{\frown}{\measuredangle}$'s and <6 r't $\overset{\frown}{\measuredangle}$'s.

568. Corollary II to 566. Every $\overset{\frown}{\measuredangle}$ of a $\overset{\frown}{\triangle}$ is $>\frac{1}{2}e$.

569. Corollary III to 566. A spherical polygon is equivalent to half its spherical excess.

Ex. 597. If a spherical angle adjacent to one angle of a spherical triangle is equal to a second angle of the triangle, the sides opposite these are together a ray.

Ex. 598. In a spherical triangle and the spherical triangle determined by the opposites of its vertices the sides and angles are respectively congruent.

Ex. 599. Where are the vertices of spherical triangles on a given base the sum of whose other sides is a ray?

Ex. 600. Does a triangle ever coincide with its polar?

Ex. 601. The difference of any two angles of a spherical triangle cannot exceed the supplement of the third.

Ex. 602. The bisector of an angle passes through the pole of the bisector of the supplemental adjacent angle.

Ex. 603. If two straightests make equal angles with a third, the sects from their poles to its are equal.

Ex. 604. If a straightest be through the pole of a second, so is the second through a pole of the first.

Ex. 605. If two circles be tangent, the point of contact is on their center-straightest.

Ex. 606. The common secant of 2 intersecting $\odot$s bisects a common tangent.

Ex. 607. The three common secants of 3 $\odot$s which intersect each other are copunctal.

Ex. 608. If a quad' can have a $\odot$ inscribed in it, the sums of the opposite sides are $=$.

Ex. 609. If two equal $\odot$s intersect, each contains the orthocenters of $\overset{\frown}{\triangle}$s inscribed in the other on the common chord as base.

Ex. 610. Three equal $\odot$s intersect at a point H, their other points of intersection being A, B, C. Show that

H is the orthocenter of $\widehat{\triangle}ABC$; and that the $\widehat{\triangle}$ formed by the centers of the circles is $\equiv$ to $\widehat{\triangle}ABC$.

Ex. 611. The feet of $\perp$s from A of $\triangle ABC$ on the external and internal bi's of $\measuredangle$s B and C are co-st' with the bisection-points of b and c. Does this hold for $\widehat{\triangle}$?

Ex. 612. (Bordage.) The centroids of the 4 $\widehat{\triangle}$s determined by four concyclic points are concyclic.

Ex. 613. The orthocenters of the 4 $\widehat{\triangle}$s determined by four concyclic points, A, B, C, D, are the vertices of a quad' $\equiv$ to $ABCD$. The incenters are vertices of an equiangular quad'.

Ex. 614. (Brahmegupta.) If the diagonals of a cyclic quad' are $\perp$, the $\perp$ from their cross on one side bisects the opposite side.

Ex. 615. If the diagonals of a cyclic quad' are $\perp$, the feet of the $\perp$s from their cross on the sides and the bisection-points of the sides are concyclic.

Ex. 616. If an inscribed equiangular polygon have an odd number of sides, it is equilateral.

Ex. 617. If a circumscribed equilateral polygon have an odd number of sides, it is equiangular.

Ex. 618. If one of two equal chords of a $\odot$ bisects he other, then each bisects the other.

Ex. 619. The tri-rectangular $\widehat{\triangle}$ is its own polar.

Ex. 620. All $=\widehat{\triangle}$s on the same side of the same base have their two sides bisected by the same straightest.

Ex. 621. If the base of a $\widehat{\triangle}$ be given, and the vertex variable, the straightests through the bisection-points of the two sides always pass through two fixed points.

Ex. 622. If A and A' be opposites, then $\widehat{\triangle}$s ABC, $A'BC$ are called *colunar*. A pole of the straightest bisecting AB and AC is also pole of the circum-$\odot$ of the colunar $\widehat{\triangle}A'BC$.

Ex. 623. Given b and $\alpha+\gamma-\beta$ to construct q-pole and radius of circum-$\odot$.

Ex. 624. If $\alpha+\beta=\gamma$, the q-pole of circum-$\odot$ is bisection-point of c.

Ex. 625. Two $\widehat{\triangle}$s with one $\widehat{\angle}$ the same and the opposite escribed $\odot$s =, have equal perimeters.

Ex. 626 The tangent at A to the circum-$\odot$ of $\widehat{\triangle}ABC$ makes with AB and AC $\widehat{\angle}$s whose difference $=\beta-\gamma$.

Ex. 627. The q-pole of the circum-$\odot$ of a $\widehat{\triangle}$ coincides with that of the in-$\odot$ of the polar $\widehat{\triangle}$; and the spherical radii of the 2 $\odot$s are complementary.

Ex. 628. From each $\widehat{\angle}$ of a $\widehat{\triangle}$ a $\perp$ is drawn to the straightest through the bisection-points of the adjacent sides. Prove these $\perp$s copunctal.

Ex. 629. Through each $\widehat{\angle}$ of a $\widehat{\triangle}$ a straightest is drawn to make the same $\widehat{\angle}$ with one side as the $\perp$ on the base makes with the other side. Prove these copunctal.

Ex. 630. Two birectangular $\widehat{\triangle}$s are $\equiv$ if the oblique $\widehat{\angle}$s are =, or if the sides not quadrants are =.

Ex. 631. In $\widehat{\triangle}$, if c is fixed and $\alpha+\beta=\pi$, then C is on a fixed straightest.

Ex. 632. (Joachimsthal.) If two diagonals of a complete spherical quadrilateral are quadrants, so is the third.

Ex. 633. (1) A quad' whose diagonals bisect each other (a *cenquad*) has its opposite sides =; (2) and inversely..

(3) Also its opposite $\widehat{\angle}$s =; (4) and inversely.

(5) Every straightest through this bisection-point (spherical center) cuts the quad' into = halves.

(6) Its opposite sides make = alternate $\widehat{\angle}$s with a diagonal.

(7) Inversely, a quad' with a diagonal making with each side a $\widehat{\angle}$ = to its alternate is a cenquad. (8) So is a quad with a pair of opposite sides = and making = alternate $\widehat{\angle}$s with a diagonal.

(9) Also a quad' with a pair of opposite sides =, and a diagonal making = alternate $\widehat{\angle}$s with the other sides and opposite $\widehat{\angle}$s not supplemental.

(10) From the spherical center $\perp$s on a pair of opposite sides are =.

(11) If two consecutive $\widehat{\measuredangle}$s of a cenquad are $=$, it has a circum-$\odot$.

(12) If two consecutive sides of a cenquad are $=$, it has an in-$\odot$.

(13) The polar of a cenquad is a concentric cenquad.

(14) A pair of opposite sides of a cenquad intersect on the polar of its spherical center.

(15) Any two consecutive vertices of a cenquad and the opposites of the other two are concyclic.

(16) If $ABCD$ be a cenquad, then A, B, C', D' and A', B', C, D are on $=$ $\odot$s with q-poles opposites.

Ex. 634. The sides of a $\widehat{\triangle}$ intersect the corresponding sides of its polar on the polar of their orthocenter.

Ex. 635. The sect which a $\widehat{\measuredangle}$ intercepts on the polar of its vertex equals a sect between poles of its sides.

Ex. 636. If a spherical quad' is inscribed, and another circumscribed touching at the vertices of the first, the crosses of the opposite sides of these quad's are on a straightest.

Ex. 637. The crosses of the sides of an inscribed $\widehat{\triangle}$ with the tangents at the opposite vertices are on a straightest.

CHAPTER XVI.

ANGLOIDS OR POLYHEDRAL ANGLES

570. Theorem. *The area of a spherical angle, L, is* $2r^2u$.

Proof. For we have the proportion, area of $\widehat{\measuredangle}$: area of $\frac{1}{4}$ sphere = size of $\widehat{\measuredangle}$: size of r't $\widehat{\measuredangle}$ = size of $\measuredangle$ at center : size of r't $\measuredangle$; that is,

$$L : r^2\pi = u : \tfrac{1}{2}\pi.$$
$$\therefore L = 2r^2u.$$

571. Corollary to 570 and 566. The area of a spherical triangle is the size of its spherical excess multiplied by its squared radius.

If e' is the u of e,

$$\widehat{\triangle} = e'r^2.$$

572. Corollary to 571. To find the area of a spherical polygon, multiply its spherical excess in radians by the squared radius.

573. Definition. Three or more rays, a, b, c, from the same point, V, taken in a certain order and such that no three consecutive are coplanar, determine a figure called a polyhedral angle or an *angloid*.

The common point V is the vertex, the rays a, b, c, . . . are *edges*, the angles $\measuredangle ab$ $\measuredangle bc$, . . . are *faces*,

and the pairs of consecutive faces are the *dihedrals* of the angloid.

According to the number of the rays, 3, 4, 5, . . . the angloid is called trihedral, tetrahedral, pentahedral, . . . , and in general polyhedral.

574. If a unit sphere be taken with the vertex of the angloid as center, this determines a spherical polygon whose angles are of the same size as the inclinations of the angloid's dihedrals, while the length of each side of the polygon is the size of the corresponding face-angle of the angloid.

Hence from any property of spherical polygons we may infer an analogous property of angloids.

For example, the following properties of trihedrals have been proved in our treatment of spherical triangles:

I. Trihedrals are either congruent or symmetrical which have the following parts congruent:

(1) Two face-angles and the included dihedral.

(2) Two dihedrals and the included face-angle.

(3) Three face-angles.

(4) Three dihedrals.

(5) Two pairs of dihedrals and the face-angles opposite one pair equal, opposite the other pair not supplemental.

(6) Two pairs of face-angles and the dihedrals opposite one pair equal, opposite the other pair not supplemental.

II. As one of the face-angles of a trihedral is greater than or equal to a second, the dihedral opposite the first is greater than or equal to that opposite the second, and inversely.

III. Symmetrical trihedrals are equivalent or equivalent by completion.

IV. Any two face-angles of a trihedral are together greater than the third.

V. In two trihedrals having two face-angles respectively congruent, if the third is greater in the first, so is the opposite dihedral, and inversely.

VI. In any trihedral the sum of the three face-angles is less than four right angles.

VII. In any trihedral, the sum of the three dihedrals is greater than two and less than six right angles.

In the same way, defining a polyhedral as convex when any polygon formed by a plane cutting every face is convex, we have:

VIII. In any convex polyhedral any face-angle is less than the sum of all the other face-angles.

Proof. Divide into trihedrals and apply IV repeatedly.

IX. In any convex polyhedral the sum of the face-angles is less than four right angles.

X. The three planes which bisect the dihedrals of a trihedral are costraight.

XI. The three planes through the edges and the bisectors of the opposite face-angles of a trihedral are costraight.

XII. The three planes through the bisectors of the face angles of a trihedral, and perpendicular to these faces, respectively, are costraight.

XIII. The three planes through the edges of a trihedral, and perpendicular to the opposite faces, respectively, are costraight.

XIV. If two face-angles of a trihedral are right, the dihedrals opposite are right.

Ex. 638. The face angles of any trihedral are proportional to the sides of its $\widehat{\triangle}$ on any sphere.

Ex. 639. The area of a $\widehat{\triangle}$ is to that of the sphere as its spherical excess is to 8 r't ∡s ($e' : 4\pi$).

Ex. 640. Find the angles and sides of an equilateral $\widehat{\triangle}$ whose area is $\frac{1}{4}$ the sphere.

Ex. 641. The angle-sum in a r't $\widehat{\triangle}$ is $<$ 4 r't ∡s.

Ex. 642. If one of the sects which join the bisection-points of the sides of a $\widehat{\triangle}$ be a quadrant, the other two are quadrants.

Ex. 643. Cut a tetrahedral by a plane so that the section is a ∥gm.

Ex. 644. To cut by a plane a trirectangular trihedral so that the section may equal any given $\triangle$.

Ex. 645. The base AC and the area of a $\widehat{\triangle}$ being given, the vertex B is concyclic with A' and C'.

Ex. 646. Given a trihedral; to each face from the vertex erect a perpendicular ray on the same side as the third edge; the trihedral they form is called the polar of the given one.

If one trihedral is the polar of a second, then the second is also the polar of the first.

Ex. 647. If two trihedrals are polars, the face angles of the one are supplemental to the inclinations of the corresponding dihedrals of the other.

Ex. 648. If two angles of a $\widehat{\triangle}$ be r't, its area varies as the third $\widehat{∡}$.

Ex. 649. If 1′, one *minute*, is one sixtieth of a degree, and 1″, one *second*, is one sixtieth of a minute, find the area of a $\widehat{\triangle}$ from the radius r, and the angles $\alpha = 20° \; 9' \; 30''$, $\beta = 55° \; 53' \; 32''$, $\gamma = 114° \; 20' \; 14''$. *Ans.* $0.1813 r^2$.

Ex. 650. All trihedrals having two edges common, and, on the same side of these, their third edges prolongations of elements of a right cone containing the two common edges, are equivalent.

Ex. 651. Equivalent $\overset{\frown}{\triangle}$s on the same side of the same base are between copolar = $\odot$s.

Ex. 652. Find the spherical excess of a $\overset{\frown}{\triangle}$ in degrees from its area and the radius.

Ex. 653. If any angloid whose size is 1, that is, any angloid which determines on the unit sphere a spherical polygon whose area is 1, be called a *steradian*, and all the angloids about a point be together called a *steregon*, then a stereogon contains 4π steradians.

APPENDIX I.

THE PROOFS OF THE TWO BETWEENNESS THEOREMS 16 AND 17, TAKEN FOR GRANTED IN THE TEXT.*

575. Theorem I. *If B is between A and C, and C is between A and D, then C is between B and D.*

Proof. Let A, B, C, D be on a. Through C take a straight c other than a. On c take a point E other than C. On the straight BE between B and E take F. Thus between B and F is no point of c. Now between A and F there can be no point of c, else c would (by II 4) have a point between A and B, since, by the construction of F, c cannot have a point between B and F. Thus C would be between A and B, contrary to our hypothesis that B is between A and C.

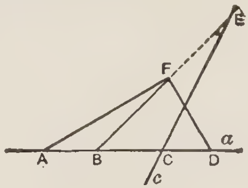

FIG. 228.

Thus since c cannot have a point between A and F, it must (by II 4) have a point between F and D. Thus we have the three non-co-straight points F, B, D, and c with a point between F and D, and, by construction, none between F and B. Therefore it must (by II 4) have a point between B and D. So C is between B and D.

* These proofs are due to my pupil, R. L. Moore, to whom I have been exceptionally indebted throughout the making of this book.

576. Theorem II. *If B is between A and C, and C is between A and D, then B is between A and D.*

Proof. Let A, B, C, D be on a. Through B take a straight b other than a. On b take a point E other than B. On the straight CE between C and E take F. Thus between C and F is no point of b. Then since by hypothesis B is between A and C, therefore b must (by II 4) have a point between A and F.

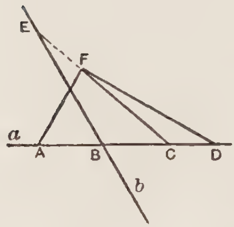

FIG. 229.

Thus we have the three non-co-straight points A, F, D, and b with a point between A and F. Therefore b must have (by II 4) a point between A and D, or between F and D. But it cannot have a point between F and D, for then it must (by II 4) have a point either between F and C, contrary to our construction, or else between C and D, contrary to Theorem I, by which C is between B and D. Therefore it has a point between A and D. So B is between A and D.

577. Theorem. III *Any four points of a straight can always be so lettered, $ABCD$, that B is between A and C and also between A and D, and furthermore C is between A and D and also between B and D.*

Proof. We may (by II 3) letter three of our points B, C, D, with C between B and D. Now as regards B and D, and our fourth point A, either A is between B and D, or B is between A and D, or D is between A and B.

If B is between A and D, we have fulfilled the hypothesis of Theorems I and II.

If D is between A and B, then interchanging the lettering for B and D, that is calling B, D and D, B, we have the hypothesis of Theorems I and II. There only remains to consider the case where A is between B and D.

If now C is between D and A, we have fulfilled the hypothesis of Theorems I and II, by calling D, A, and C, B, and A, C, and B, D.

If, however, A were between C and D we would have fulfilled the hypothesis of Theorems I and II by writing for A, B, for D, A, and for B, D.

We have left only one sub-case to consider, that where D is between A and C. This sub-case is impossible. Suppose $ABCD$ on a. Through C take a straight c other than a. On c take a point E other than C. On the straight DE between D and E take F. Thus between D and F is no point of c.

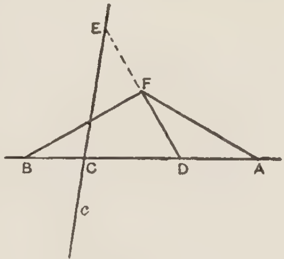

FIG. 230.

Then since by hypothesis C is between B and D, therefore c must (by II 4) have a point between B and F. Therefore we have the three non-costraight points B, F, A, and c with a point between B and F. Therefore c has (by II 4) either a point between B and A, or a point between F and A. But it cannot have a point between F and A, else it would (by II 4) either have a point between F and D, contrary to our construction, or else between D and A, giving C between D and A, contrary to our hypothesis D between A and C.

So C would be between B and A, but this with D between A and C gives (by Theorem I) D between A and B, contrary to our hypothesis A between B and D.

Thus there is always such a lettering that B is between A and C, and C between A and D, whence (by Theorem I) C is between B and D, and (by Theorem II) B is between A and D.

578. Theorem. *Are A, B, C, D points of a straight, such that C lies between A and D and B between A and C, then lies also B between A and D, but not between C and D.*

Fig. 231.

Proof. The points $ABCD$, in accordance with 577, have an order in which two are each between the remaining pair and of this remaining pair neither is between two others. But here by hypothesis C and B are between others. So we reach the following arrangements $ACBD$, $DBCA$, $ABCD$, $DCBA$. Of these arrangements, however, the first two do not satisfy the hypothesis. For in both arrangments C lies between A and B, which (by II 3) contradicts the hypothesis "B between A and C."

In the third and fourth arrangement appears, by 577, that C lies between B and D, therefore, by II 3, B cannot lie between C and D.

579. Theorem. *Between any two points of a straight there are always indefinitely many points.*

Proof. By II 2, there is between A and B at least one point C; likewise there is between A and

C at least one point C'. Further, there is within AC' at least one point C'', which likewise is within AB but not within $C'B$; therefore since C lies within $C'B$, C'' cannot be identical with C. In this way

FIG. 232.

we get ever new points of AB without ever coming to an end.

580. Theorem. If $ABCD$ is an arrangement of four points corresponding to 577, then there is besides this arrangement only still the inverse which fulfills 577. [The proof is essentially already given in proving 578.]

581. Theorem. If any finite number of points of a straight are given, then they can always be arranged in a succession $A, B, C, D, E, \ldots, K$, such that B lies between A on the one hand and $C, D, E, \ldots, K$ on the other, further C between A, B on one side and $D, E, \ldots, K$ on the other, then D between A, B, C on the one side and $E, \ldots, K$ on the other, and so on.

Besides this distribution there is only one other, the reversed arrangement, which is of the same character.

[This theorem is a generalization of 577.]

FIG. 233.

Proof. Our theorem holds for four points by 577 and 580.

We may show that the theorem remains valid for $n+1$ points if it holds for n points.

Let $A_1A_2A_3 \ldots A_n$ be the desired arrangement for n points. If further we take an additional point then there are forthwith three cases possible:

(1) A_1 lies between X and A_n;

(2) A_n lies between X and A_1;

(3) X lies between A_1 and A_n.

In the third case we prove further, that there is one and only one number m, such that X lies between A_m and A_{m+1}.

Finally we show that the following arrangements in the three cases have the desired properties:

(1) $XA_1A_2A_3 \ldots A_n$;

(2) $A_1A_2A_3 \ldots A_nX$;

(3) $A_1A_2A_3 \ldots A_mXA_{m+1} \ldots A_n$;

and that they with their inversions are the only ones which possess those properties.

APPENDIX II.

THE COMPASSES.

582. Euclid's third postulate is: *About any center with any radius one and only one circle may be taken.* This has been understood in ordinary geometries as authorizing the use of a physical instrument, the compasses, for drawing a circle with any center and any radius.

But this is only made fruitful, beyond the sect-carrier, in problem solving, by two new assumptions:

Assumptions of the Compasses.

Assumption VI 1. *If a straight have a point within a circle, it has two points on the circle.*

Assumption VI 2. *If a circle have a point within and a point without another circle, it has two points on this other.*

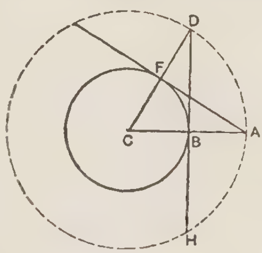

FIG. 234.

583. Problem. From a given point without the circle to draw a tangent to the circle.

Construction. Join the given point A with the center C, meeting the circle in B. Erect $BD\perp$ to CB, and (by VI 1) cutting in

D the $\odot C(CA)$. Join DC, meeting $\odot C(CB)$ in F. Then AF is tangent to $\odot C(CB)$.

Proof. Radius CA, $\perp$ to chord HD, bisects arc HD; $\therefore$ if we rotate the figure until H comes upon the trace of A, then A is on the trace of D; $\therefore$ tangent HB on trace of AF.

Determination. Always two and only two tangents.

584. Problem. To construct a triangle of which the sides shall be equal to three given sects, given that any two whatever of these sects are together greater than the third.

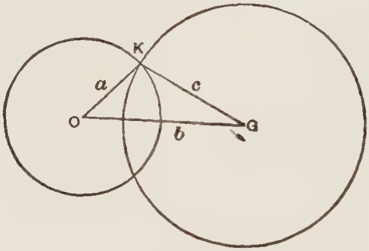

FIG. 235.

Given the three sects a, b, c, any two whatever together greater than the third.

Construction. On a straight OF from O take $OG=b$. Take $\odot O(a)$, and $\odot G(c)$. Since $a+c>b$, these (by VI 2) intersect, say at K.

$\triangle OGK$ is the triangle required.

585. Problem. To construct a triangle, given two sides and the angle opposite one of them.

Given a, c, and C.

CASE 1. If $a<c$.

On one ray of $\measuredangle C$ take $CB=a$. Take $\odot B(c)$. This (by VI 1) has two points A', A_1 on the straight of the other ray of $\measuredangle C$. The point C is between A' and A_1. $\therefore$ if $\measuredangle C$ r't, we have two congruent triangles (Fig 236); if oblique, only one triangle (Fig. 237).

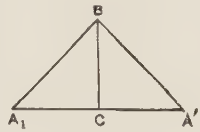

FIG. 236.

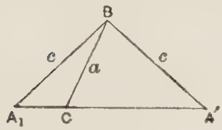

FIG. 237.

CASE 2. If $a=c$. [$\measuredangle C$ acute.] Then C coincides with A' or A_1, and we have only one triangle.

CASE 3. If $a>c$. [$\measuredangle C$ acute.]

I. If $c=p$, the perpendicular from B on CA, there is only one triangle.

II. If $c>p$, then A_1 and A' are on the same side of C and there are two different triangles which fulfill the conditions, namely, $A'BC$ and A_1BC (Fig. 238).

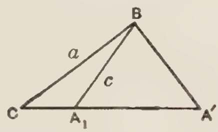

FIG. 238.

This is called the *ambiguous case.*

III. If $c<p$, no triangle.

APPENDIX III.

THE SOLUTION OF PROBLEMS.

586. A problem in geometry is a proposition asking for the graphic construction of a figure which shall satisfy or fulfill certain given conditions or requirements. It has been customary to use the ruler and compasses; that is, to allow our assumptions I–V and also VI (Appendix II), but no others. Of these, assumptions V–VI have usually been superfluous and unnecessary, the problems treated not requiring the compasses, but only ruler and sect-carrier.

587. When we know how to solve a problem, the treatment consists of

(1) Construction: Indicating how the ruler and sect-carrier or ruler and compasses are to be used in effecting what is required.

(2) Proof: Showing that the construction gives a figure fulfilling all the requirements.

(3) Determination: Considering the possibility of the solution, and fixing whether there is only a single solution or suitable result of the indicated procedure, or more than one, and discussing the limitations which sometimes exist, within which alone the solution is possible.

588. The first step toward finding a desired solution is usually what is called Geometrical Analysis.

This consists in supposing drawn a figure like the one desired, also containing the things given, and then analyzing the relations of the given things among themselves and to the things or figure sought, or the elements necessary for attaining such figure.

589. Methods of procedure in problem-solving.

I. *Successive Substitutions.*—We may substitute for the required construction another from which it would follow, and for this another, perhaps simpler, until one is reached which we know how to accomplish.

Just so, in attempting to find a demonstration for a new theorem, we may freely deduce from the desired proposition by use of invertible theorems, and if thus we reach a known proposition, the inversion of the process will give the demonstration sought.

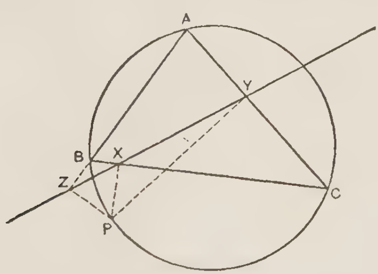

FIG. 239.

I Example 1. Theorem. If from any point P on the circumcircle of the triangle ABC be drawn PX,

PY, PZ perpendicular to the sides, the points X, Y, Z will be costraight, the Simson's st' of $\triangle$ for P.

Analysis. We will have proven X, Y, Z costraight if on joining XY, XZ we show $\measuredangle PXY$ supplementary to $\measuredangle PXZ$. But again this is proven if we show $\measuredangle PXZ$ supplementary to $\measuredangle ABP$ and $\measuredangle PXY \equiv \measuredangle ABP$.

But $\measuredangle PXZ$ is supplement of $\measuredangle ABP$ since P, X, Z, B are concyclic. And since P, Y, C, X are concyclic, $\measuredangle PXY$ is supplement of $\measuredangle PCY$, as is also $\measuredangle ABP$, since P, C, A, B are concyclic.

I Example 2. Problem. Construct a $\triangle$, given an angle, the side opposite and two sects proportional to the other two sides. [$\triangle$ from a, α, b/c.]

Analysis. By a and α is (by 165) the circumcircle given. Bisect $\measuredangle BAC$ by AD and prolong AD to meet the circle again (by 138) in E. Then (by 242) $CD/BD = b/c$. So (by 241) the point D is known. Moreover, since arc $BE =$ arc CE, the point E is (by 225) known. Therefore, taking $BC = a$, the points D and E can be constructed, and thus A by the prolongation of ED to the circle.

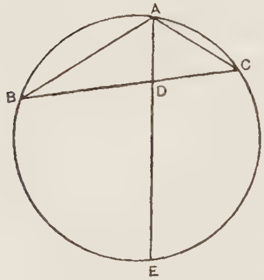

Fig. 240.

Analogous Problems: Ex. 1. Given a, R, b/c. Ex. 2. Given α, R, b/c. Ex. 3. Given α, BD, CD.

II. *Data.*—The explicit giving of certain things may involve the implicit giving of others more immediately available.

Such an implicitly given thing has been called a *datum*.

For example, if a straight and a point without it be given, the perpendicular from the point to the straight is a datum.

If an angle and a point within it be given, then the sect from the point to the vertex, the sect from the point to one side drawn parallel to the other side, and the sect this cuts off from the side are data.

With an angle α are given the constructible parts $\frac{1}{2}\alpha$, $\frac{1}{4}\alpha$, etc., but not $\frac{1}{3}\alpha$, $\frac{1}{6}\alpha$, etc.; also the supplement and complement.

If the sum and difference of two magnitudes are given so are the magnitudes.

If in a triangle of the three things, a side, the opposite angle, the circumradius, two are given, so is the third.

So also with base, altitude, area.

II Example 1. To construct a triangle from one side, the opposite angle, and the difference of the other two angles. ($\triangle$ from a, α, $\beta-\gamma$.)

Analysis. Since α is known, so is also $\beta+\gamma$ as its supplement. $\therefore$ β and γ are known, and we have a side and the two adjoining angles.

III. *Translation.*—Again new auxiliary parts may advantageously be introduced. Certain procedures are found particularly fertile.

In any triangle ABC, transporting AC parallel to itself into BD, and extending BA equal to itself to E, we have that the sides of EDC are double the medians of ABC and parallel to them.

The sides of ABC are two-thirds the medians of EDC, and A is its centroid. Two of the altitudes of the triangles AED, AEC, ADC are equal to two altitudes of ABC and the content of EDC is triple that of ABC.

If we have given such elements of ABC as render possible, through these properties, the determina-

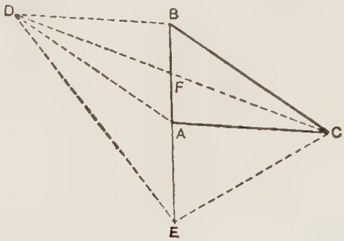

FIG. 241.

tion of one of the triangles EDC, AED, ADC, AEC, then the triangle ABC will always be constructible.

III Example 1. Problem. Construct a triangle, given one median m_1, the angle between the others, m_2 and m_3, and two sects proportional to them:

$$[\triangle \text{ from } m_1, \measuredangle \text{of } m_2 \text{ and } m_3, m_2/m_3].$$

Analysis. In the $\triangle EDC$ we know $DC = 2FC = 2m_1$; also $\measuredangle E$ and DE/EC. Therefore the problem is reduced to I Example 2.

IV. *Symmetry.*—Add to the tentatively constructed figure the figure symmetrical to it or to a part of it, with respect to a chosen straight as axis. Particularly adapted for axis is an angle-bisector or a perpendicular. Especially does this show differences of sects or angles.

IV Example 1. Problem. To construct a triangle, given two sides and the difference of the opposite angles:

$[\triangle$ from $a, b, \alpha-\beta]$.

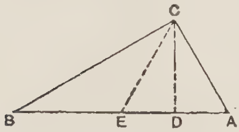

FIG. 242.

Analysis. Take CE ⫲ CA with respect to perpendicular CD. Then $\measuredangle AEC=\alpha$. But $\measuredangle AEC=\measuredangle B+\measuredangle BCE$ $=\beta+\delta$. $\therefore\ \delta=\alpha-\beta$.

$\triangle BCE$ is therefore constructible (two sides and included $\measuredangle$).

The point D is the foot of the perpendicular from C on BE. A ⫲ E (axis CD).

IV Example 2. Given the two straights s and l and a point A on s. Determine the point X on s, such that $AX=$ perpendicular from X on l.

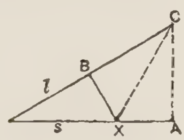

FIG. 243.

Analysis. With respect to the bisector of $\measuredangle AXB$ as axis, the figure symmetrical to XBC is XAC, and $\measuredangle ACB$ is bisected by XC. Therefore, erecting a perpendicular at the given point A, the bisector of the angle ACB made with l meets s in the required point X.

V. *Similarity.*—When the figure to be constructed is determinable by such conditions that, omitting one, the remaining determine a system of similar figures, then first determine this system of figures similar to the one sought, and secondly, by taking account of the suppressed condition, fix the one which solves the problem.

There are two sorts of cases according as the suppressed condition determines (I) the size of the figure sought; (II) its position or place. In the first case the data are angles and proportions (giving the system of similar figures), and a sect (giving the size of the particular figure). In the second case are only given angles and proportions, but also is imposed a condition that the figure demanded must have a determinate position with respect to something given; for example, must contain a given point, must contain a tangent to a given circle, etc.

V Example 1. To make a triangle, given an angle α, sects proportional to the sides containing it ($b:c=m:n$), and also its bisector t_1:

$$[\triangle \text{ from } \alpha,\ b/c,\ t_1].$$

On the sides of α take m and n. On its bisector take t_1. Through the foot of t_1 draw a parallel to the straight through the ends of m and n. [Similarly: $\triangle$ from α, β, m_a; $\triangle$ from α, β, h_a].

V Example 2. In a triangle ABC inscribe a parallelogram of which one angle shall coincide with angle BAC, and such that its sides are as m to n.

The vertices of the parallelogram sought must be one at A, one on b, one on c, one on a. Omitting this last condition, the fourth vertices of parallelograms satisfying the other conditions are on a straight determined by A and the parallelogram with sides m and n.

The fourth vertex sought is where this straight crosses the base b.

VI. *Intersection of Loci.*—All points in a plane which satisfy a single geometric condition make up often a single straight or a single circle, in rare cases more than one.

Neglecting these rare cases, we may call such straight or circle the *locus* (place) of the points satisfying the given condition.

Where it is required to find points satisfying two conditions, if we leave out one condition, we may find a locus of points satisfying the other condition.

Thus, for each condition we may construct the corresponding locus. If these two loci have points in common, these points, and these only, satisfy both conditions.

In a problem involving more than two distinct conditions, two may be selected which give available loci, and then the remaining used to complete the solution. If the circle occurs as locus, we may assume the two postulates of the compasses (VI).

As preliminary it will be convenient to have a collection of simple loci.

LOCI.

1. The locus of points which with a given point P give the sect r is $\odot P(r)$.

This is also the locus of the centers of circles with radius r, which pass through P.

2. The locus of points P on one side of a st' l and such that from them the perpendiculars on l are equal to h is the parallel to l through P, one such point.

This is also the locus of the centers of circles with radius r tangent to l on one side.

3. The locus of the point to which sects from two given points are equal is the perpendicular bisector of the sect joining them.

This is also the locus of the centers of all circles through the two given points.

4. The locus of the vertices of all right angles on the same side of a given sect as hypothenuse is the semicircle on the given sect as diameter.

5. The locus of the vertices of all angles congruent to β on a given sect AC as base is the arc on AC having β as inscribed angle.

This is the locus of the vertex of triangles on the same side of given base b with given opposite angle β.

6. The locus of the point from which perpendiculars on the sides of a given angle are equal is the angle-bisector.

This is also the locus of the centers of circles touching both sides of the angle from within.

7. The locus of the bisection-points of all chords equal to k in a given circle is a concentric circle with radius equal to the perpendicular from the center on k.

8. The locus of the bisection-points of all chords of a circle through a given point P is the circle on the sect from P to the given center as diameter.

9. The locus of the centers of circles touching a given st' l at the point P is the perpendicular to l through P.

10. The locus of the centers of circles touching the given circle $\odot C(CP)$ at P is the st' CP.

11. The locus of the centers of circles with radius r_1 touching a given circle with radius r_2 from without is a concentric circle with radius r_1+r_2. From within, radius r_2-r_1.

12. The locus of the end-points of tangents $=t$ to the circle $\odot C(r)$ is the concentric circle with radius from C to the end of one of these tangents.

This circle is also the locus of the centers of circles with radius t which cut $\odot C(r)$ at right angles; that is, so that at every intersection-point the tangents to the two circles are at right angles.

13. The locus of the centers of circles of radius r with centers on the same side of st' l and cutting from l a chord $=k$ is a parallel to l through the vertex of an isosceles triangle with base k on l and side r.

14. The centers of circles of radius r_1 cutting from a given circle $\odot C(r_2)$ an arc with chord $=k$ lie on two concentric circles with radius from C to the vertex of an isosceles triangle with base k a chord of $\odot C(r_2)$ and side r_1.

15. The locus of the centers of circles with radius r_1 which bisect a given circle $\odot C(r_2)$ is a concentric circle with radius from C to the vertex of an isosceles triangle with base a diameter of $\odot C(r_2)$ and side r_1.

16. The locus of the centers of circles with radius r_1 which are bisected by a given circle $\odot C(r_2)$ is a concentric circle having as radius the perpendicular from C on any chord $=2r_1$ in $\odot C(r_2)$.

17. The locus of the vertices of triangles of equal content on the same side of the same base is the parallel to the base through the top of its altitude.

18. The locus of the vertex B of all triangles on the same base b in which $a^2+c^2=$constant is a circle.

19. The locus of the vertex B of all triangles on the same base b in which $a^2-c^2=$constant is a perpendicular to the base.

20. The locus of the bisection-point of a sect with end-points on two straights at right angles is a circle with their intersection as center and half the sect as radius.

VI Example 1. To construct a triangle from an angle and the altitudes on the including sides ($\triangle$ from α, h_2, h_3).

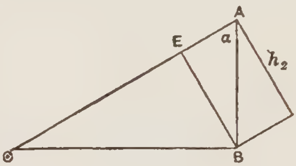

FIG. 244.

Analysis. B is the intersection of a side of α with the parallel to the other side of α through the end of a perpendicular to this side equal to h_2.

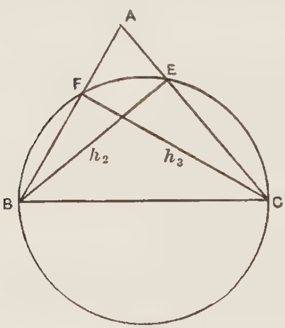

FIG. 245.

VI Example 2. To construct a triangle from one side and the altitudes to the other two ($\triangle$ from a, h_2, h_3).

Analysis. E is the intersection of semicircle on BC with arc of radius $=h_2$. So for F. Then A is intersection of CE and BF.

VII. *Reckoning.* — Our sect calculus may be freely used for making and solving equations of the first and second degree containing expressions for sought sects.

VII Example 1. Construct, without using the compasses, an isosceles triangle with the equal angles double the third.

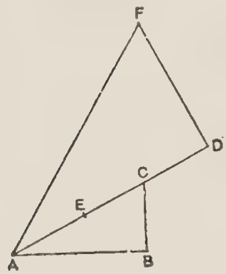

FIG. 246.

Construction. On one side of a right angle, B, take BA equal to half the unit sect. $BA = \frac{1}{2}$. On the other side, $BC = \frac{1}{4}$. Prolong the hypothenuse AC to D, taking $CD = \frac{1}{4}$. At D erect a perpendicular, and on it take $DF = \frac{1}{2}$.

On CA take $CE = \frac{1}{4}$. Take $AA' = 2AE$. At E erect the perpendicular bisector EG, taking $EG = AF$. Join AG and $A'G$. The triangle $AA'G$ is the one required.

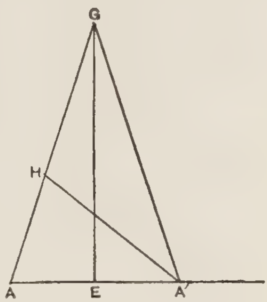

FIG. 247.

Proof. Take $GH = AA'$. Join $A'H$.

$$EG = AF = \left[\left[\left(\tfrac{1}{4} + \tfrac{1}{16}\right)^{\frac{1}{2}} + \tfrac{1}{4}\right]^2 + \tfrac{1}{4}\right]^{\frac{1}{2}}$$

$$= \left[\left[\tfrac{1}{4}\left[(5)^{\frac{1}{2}} + 1\right]\right]^2 + \tfrac{1}{4}\right]^{\frac{1}{2}}$$

$$= \left[\tfrac{1}{16}\left[10 + 2(5)^{\frac{1}{2}}\right]\right]^{\frac{1}{2}}.$$

$$AE = \tfrac{1}{4}\left[(5)^{\frac{1}{2}} - 1\right].$$

$$\therefore AG^2 = \tfrac{1}{16}\left[10 + 2(5)^{\frac{1}{2}}\right] + \tfrac{1}{16}\left[6 - 2(5)^{\frac{1}{2}}\right] = 1.$$

$$GH = AA' = 2AE = \tfrac{1}{2}\left[(5)^{\frac{1}{2}} - 1\right].$$

$$AH = 1 - \tfrac{1}{2}\left[(5)^{\frac{1}{2}} - 1\right] = \tfrac{1}{2}\left[3 - (5)^{\frac{1}{2}}\right].$$

$$\therefore GA : AA' = AA' : AH.$$

$$\therefore \text{(by 239)} \ \triangle AGA' \sim \triangle AA'H.$$

$$\therefore A'H = AA' = GH.$$

$$\therefore \measuredangle AA'G = \measuredangle A'AG = \measuredangle AHA' =$$
$$= \measuredangle HGA' + \measuredangle HA'G = 2 \measuredangle HGA'.$$

VIII. *Partition of a Perigon.*—The four right angles around a point, taken together, may be called a *perigon*. Since any angle may be bisected, a perigon can be cut into 2^n congruent angles.

Since the supplement of the angle of an equilateral triangle is one-third of four right angles, therefore a perigon can be cut into $3 \cdot 2^n$ congruent angles.

The angles at the base of an isosceles triangle with the equal angles each double the third are each one-fifth of four right angles. Therefore a perigon can be cut into $5 \cdot 2^n$ congruent angles.

The difference between one-third and one-fifth of a perigon is two-fifteenths of a perigon. Hence a perigon can be cut into $15 \cdot 2^n$ congruent angles.

If a perigon be cut into n congruent angles, the rays determine on any circle about the vertex the vertices of an inscribed regular polygon of n sides, and the points of tangency of a regular circumscribed polygon of n sides.

From the time of Euclid, about 300 B.C., no advance was made in the inscription of regular polygons until Gauss, in 1796, found that a regular polygon of 17 sides was inscriptible, and in 1801 published the following:

That the geometric division of the circle into n equal parts may be possible it is necessary and sufficient that n be 2 or a higher power of 2, or else a prime number of the form $2^{2^m} + 1$, or a product of two or more different prime numbers of that form,

or else the product of a power of 2 by one or more different prime numbers of that form. Below 300, the following 38 are the only possible values of n: 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272.

There is only one inscriptible regular polygon known with the number of its sides prime and greater than 257. This number is $2^{2^4}+1=65{,}537$. For $m=5$, $m=6$, $m=7$, the numbers obtained are not prime. Further no one has gone.

Ex. 654. Show how to trisect the central $\measuredangle$, the interior $\measuredangle$, and the exterior $\measuredangle$ of a regular n-gon where $n=2^m$ or 5.2^m.

Ex. 655. Show how to cut the $\measuredangle$ o an equilateral $\triangle$ into n equal parts if n is 2^m or 3.2^m or 5.2^m.

LOCI.

Ex. 656. In $\triangle$, given b and β, find the locus of G; H; I; I_1; O.

Ex. 657. In $\odot C(r)$ find the locus of C:

(1) Given r and P (a point of $\odot$).
(2) Given r and a tangent.
(3) Given P, Q (two points of $\odot$).
(4) Given tangent at P.
(5) Given 2 $\parallel$ tangents.

Ex. 658. In $\triangle$, given b and β, find locus of the bisection-point of sect joining outer vertices of equilateral $\triangle$s on a and c.

Ex. 659. Find locus of point the sum of the squares of whose sects to A, B, $C=k$.

Ex. 660. Given an equilateral $\triangle$, find the locus of the point whose sect to one vertex is the sum of its sects to the others.

Ex. 661. Find the locus of the point the sum of the squares of whose $\perp$s to the sides of a r't $\measuredangle$ $=k^2$.

Ex. 662. Find the locus of the intersection of two secants drawn through the ends of a fixed diameter in a given ⊙, one of the secants ⊥ to a tangent at the second point where the other cuts ⊙.

Ex. 663. Find the locus of the intersection of 2 st's drawn from the acute ∡s of a r't △, through the points where any ⊥ to hypothenuse cuts one opposite side and the production of the other.

Ex. 664. Two given ⊙s intersect. Find the locus of the bisection-point of the sect through one of their points of intersection with end-points one on each circle.

Ex. 665. Given AB divided at C. Find locus of P, if $\measuredangle APC = \measuredangle BPC$.

Ex. 666. Any 2⊥ chords intersect in a given point of a given ⊙. Find the locus of the bisection-point of a chord joining their ends.

Ex. 667. The locus of a point, the sum of the squares of whose sects from the vertices of a given equilateral △ equals twice the square on one of the sides, is the circum-⊙.

Ex. 668. The locus of the end of a given sect from the point of contact and on the tangent is a concentric ⊙.

Ex. 669. Find the locus of the foot of the ⊥ from P on a st' through B.

Ex. 670. Find the locus of the end of sect from P cut by st' a into parts as m to n.

Ex. 671. Find locus of end of sect from st' a cut by P into parts as m to n.

Ex. 672. Sects ∥ and with ends in the sides of $\measuredangle\alpha$ are cut into parts as m to n. Find the locus of the cutting points.

Ex. 673. Find the locus of a point P if $PA : PB = m : n$.

Ex. 674. The locus of the cross of two tangents to $\odot C(r)$, the st' of whose chord of contact rotates about a fixed point P is a st' $p \perp CP$.

P is called the pole of p, and p the polar of P, with respect to the given ⊙.

Ex. 675. If A (given) is on the polar of X (variable), find the locus of X.

Ex. 676. To find the locus of a point from which rays through the ends of a given sect make a given ∡.

Ex. 677. Find the locus of the vertex B of △s, given b and $a:c=m:n$.

Ex. 678. [Circle of Apollonius.] If a sect is cut into parts as m to n, and the interior and exterior points of division are taken as ends of a diameter, this ⊙ contains the vertices of all △s on the given sect, whose other two sides are as m to n.

Ex. 679. Find the locus of those points in a plane α from which rays to the ends of a given sect not on α are ⊥.

Ex. 680. Find the locus of P if $PA=PB=PC$.

Ex. 681. If b' is the projection of b on $\alpha \| b$, and $b' \perp a$ in α, find the locus of the bisection-point of a given sect AB if A on a and B on b.

Ex. 682. A variable st' is ∥ to a given plane and meets two non-coplanar st's. Find the locus of a point which cuts the intercepted sect into parts as m to n.

Ex. 683. Find the locus of the point from which ⊥s to three coplanar st's are =.

Ex. 684. Find the locus of the point having one or two of the following:

(I) Equal sects to two given points;

(II) Equal ⊥s to two given intersecting st's;

(III) Equal ⊥s to two given planes.

Ex. 685. Find the locus of the poles of great circles making a given angle with a given great circle.

Ex. 686. Calling 2 ∡s *complemental* when their sum is a r't ∡, what is the locus of the intersection of rays from A and B making ∡ with AB the complement of ∡ with BA?

Ex. 687. Calling a chord the *chord of contact of* the point of interesection of tangents at its extremities, what is the locus of points whose chords of contact in $\odot C(r)$ equal r?

Ex. 688. The locus of vertex of $\triangle = s^2$, on given b, is st' ∥ b at altitude h_b, where $bh_b = 2s^2$.

Ex. 689. The locus of vertex of $\triangle$ on given b, and with $a^2-c^2=s^2$, is st' $\perp$ to b at D, where $AD^2-CD^2=s^2$.

Ex. 690. Find locus of trisection points of equal chords.

Ex. 691. The locus of a point from which tangents to two given $\odot$s are $=$ is a st' $\perp$ to the center sect, which so divides it that the difference of the sq's of the segments $=r^2-r_1^2$. This st' is the *radical axis* of the 2$\odot$s. If they intersect it contains their common chord.

Ex. 692. The locus of P such that $PA:PB=m:n$ is $\odot$ on D_1D_2 as diameter, where DAB co-st' and $DA:DB=m:n$.

Ex. 693. Given b and $a-c$, the locus of foot of $\perp$ from A on t_b is $\odot$ with bisection-point of b for center and $\frac{1}{2}(a-c)$ for radius.

Ex. 694. Given b and $a+c$, the locus of foot of $\perp$ from A on bisector of external $\measuredangle$ at B is $\odot$ with bisection-point of b for center and $\frac{1}{2}(a+c)$ for radius

Ex. 695. The locus of P cutting sects from A to a as m to n is a st' $\parallel a$.

Ex. 696. The locus of P cutting a sect s from A to a so that $s\cdot AP=k^2$ is a $\odot$.

Ex. 697. The locus of P cutting a sect from $\odot C(r)$ to A as m to n is a $\odot$.

Ex. 698. If rectangles have one vertex at A and the adjacent vertices on $\odot C(r)$, the locus of the fourth vertex is $\odot C(r_1)$ where $r_1^2=2r^2-AC^2$.

Ex. 699. The locus of the vertex B of a $\widehat{\triangle}$ of given b and area is arc $A'BC'$.

Ex. 700. Given b and $(\alpha+\gamma-\beta)$ in $\widehat{\triangle}$, the locus of B is arc ABC.

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